

- To formalise the notion of a sound argument, we need some extra terminology...
- **Definition:** If  $\phi \in \mathcal{W}$ , then:

1. if there is *some* interpretation  $\pi$  such that

 $\pi \models \phi$ 

then  $\phi$  is said to be *satisfiable*, otherwise  $\phi$  is *unsatisfiable*. 2. if

 $\pi \models \phi$ 

for *all* interpretations  $\pi$ , then  $\phi$  is said to be *valid*.

• Valid formulae of propositional logic are called *tautologies*.

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## • Definition: If

 $\{\phi_1,\ldots,\phi_n,\phi\}\subseteq\mathcal{W}$ 

then  $\phi$  is said to be a *logical consequence* of  $\{\phi_1, \ldots, \phi_n\}$  iff  $\phi$  is satisfied by all interpretations that satisfy

 $\phi_1 \wedge \cdots \wedge \phi_n$ .

• We indicate that  $\phi$  is a logical consequence of  $\phi_1, \ldots, \phi_n$  by writing

$$\{\phi_1,\ldots,\phi_n\}\models\phi$$

• An expression like this is called a *semantic sequent*.

• Theorem:

If *φ* is a valid formula, then ¬*φ* is unsatisfiable;
If ¬*φ* is unsatisfiable, then *φ* is valid.

• We indicate that a formula  $\phi$  is valid by writing

 $\models \phi$ .

• We can now define the *logical consequence*.

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• Theorem:

## $\{\phi_1,\ldots,\phi_n\}\models\phi.$

iff

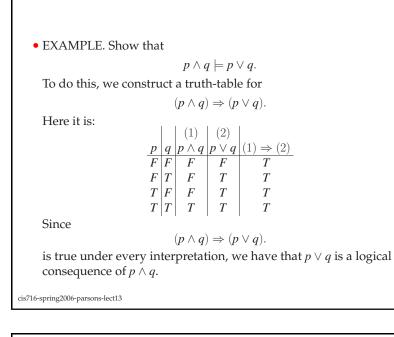
$$\models (\phi_1 \wedge \cdots \wedge \phi_n) \Rightarrow \phi.$$

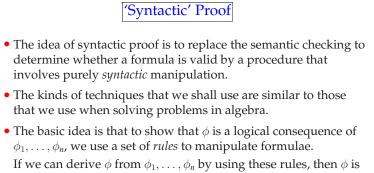
- So we have a method for determining whether  $\phi$  is a logical consequence of  $\phi_1, \dots, \phi_n$ : we use a truth table to see whether  $\phi_1 \wedge \dots \wedge \phi_n \Rightarrow \phi$  is a tautology. If it is, then  $\phi$  is a logical consequence of  $\phi_1, \dots, \phi_n$ .
- Our main concern in proof theory is thus to have a technique for determining whether a given formula is valid, as this will then give us a technique for determining whether some formula is a logical consequence of some others.

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If we can derive  $\phi$  from  $\phi_1, \ldots, \phi_n$  by using these rules, then  $\phi$  is said to be *proved* from  $\phi_1, \ldots, \phi_n$ , which we indicate by writing

$$\phi_1,\ldots,\phi_n\vdash\phi$$

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- The notion of logical consequence we have defined above is acceptable for a *definition* of a sound argument, but is not very helpful for checking whether a particular argument is sound or not.
- The problem is that we must look at all the possible interpretations of the primitive propositions. While this is acceptable for, say, 4 primitive propositions, it will clearly be unacceptable for 100 propositions, as it would mean checking 2<sup>100</sup> interpretations.

(Moreover, for first-order logic, there will be an *infinite* number of such interpretations.)

• What we require instead is an alternative version of logical consequence, that does not involve this kind of checking. This leads us to the idea of *syntactic* proof.

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- The symbol  $\vdash$  is called the *syntactic turnstile*.
- An expression of the form

$$\phi_1,\ldots,\phi_n\vdash\phi.$$

is called a *syntactic sequent*.

• A rule has the general form:

$$\frac{\vdash \phi_1; \cdots; \vdash \phi_n}{\vdash \phi}$$
 rule name

Such a rule is read:

If  $\phi_1, \ldots, \phi_n$  are proved then  $\phi$  is proved.

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• EXAMPLE. Here is an example of such a rule:

 $\begin{array}{c} \vdash \phi; \vdash \psi \\ \hline \vdash \phi \wedge \psi \end{array} \wedge \text{-I} \end{array}$ 

This rule is called *and introduction*. It says that if we have proved  $\phi$ , and we have also proved  $\psi$ , then we can prove  $\phi \wedge \psi$ .

• EXAMPLE. Here is another rule:

$$\frac{\vdash \phi \land \psi}{\vdash \phi; \vdash \psi} \land \textbf{-E}$$

This rule is called *and elimination*. It says that if we have proved  $\phi \land \psi$ , then we can prove both  $\phi$  and  $\psi$ ; it allows us to eliminate the  $\land$  symbol from between them.

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- It should be clear that the symbols ⊢ and ⊨ are related. We now have to state exactly *how* they are related.
- There are two properties of ⊢ to consider:
  - soundness;
  - completeness.
  - Intuitively, ⊢ is said to be *sound* if it is correct, in that it does not let us derive something that is not true.
  - Intuitively, *completeness* means that ⊢ will let us prove anything that is true.

- Let us now try to define precisely what we mean by *proof*.
- **Definition:** (Proof) If

$$\{\phi_1,\ldots,\phi_n,\phi\}\subseteq\mathcal{W}$$

then there is a proof of  $\phi$  from  $\phi_1, \ldots, \phi_n$  iff there exists some sequence of formulae

 $\psi_1,\ldots,\psi_m$ 

such that  $\psi_m = \phi$ , and each formula  $\psi_k$ , for  $1 \le k < m$  is either one of the formula  $\phi_1, \ldots, \phi_n$ , or else is the conclusion of a rule whose antecedents appeared earlier in the sequence.

• If there is a proof of  $\phi$  from  $\phi_1, \ldots, \phi_n$ , then we indicate this by writing:

 $\phi_1,\ldots,\phi_n\vdash\phi.$ 

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• **Definition:** (Soundness) A proof system  $\vdash$  is said to be *sound* with respect to semantics  $\models$  iff

 $\phi_1,\ldots,\phi_n\vdash\phi$ 

implies

$$\phi_1,\ldots,\phi_n\models\phi.$$

• **Definition:** (Completeness) A proof system ⊢ is said to be *complete* with respect to semantics ⊨ iff

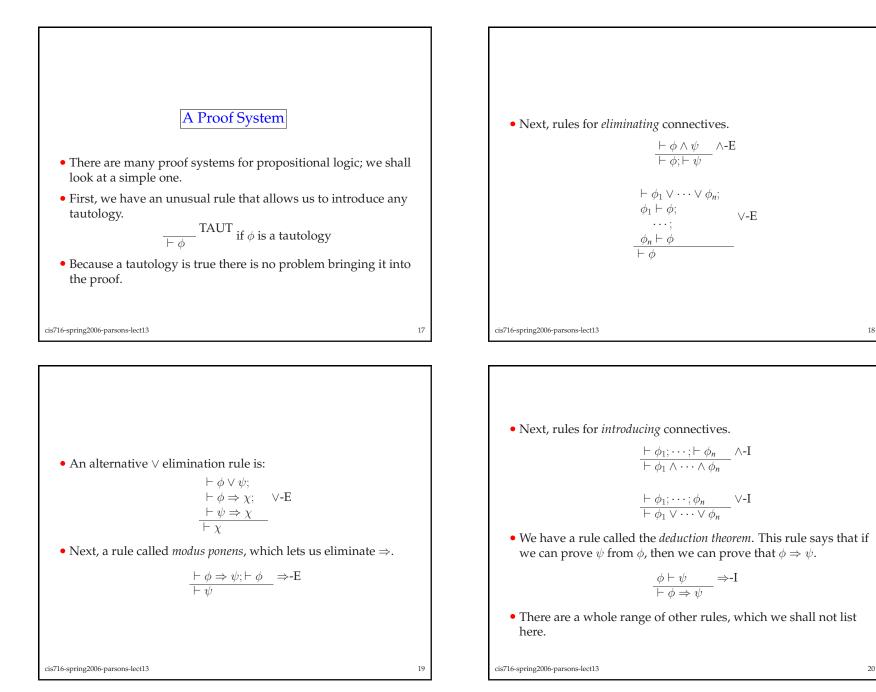
 $\phi_1,\ldots,\phi_n\models\phi$ 

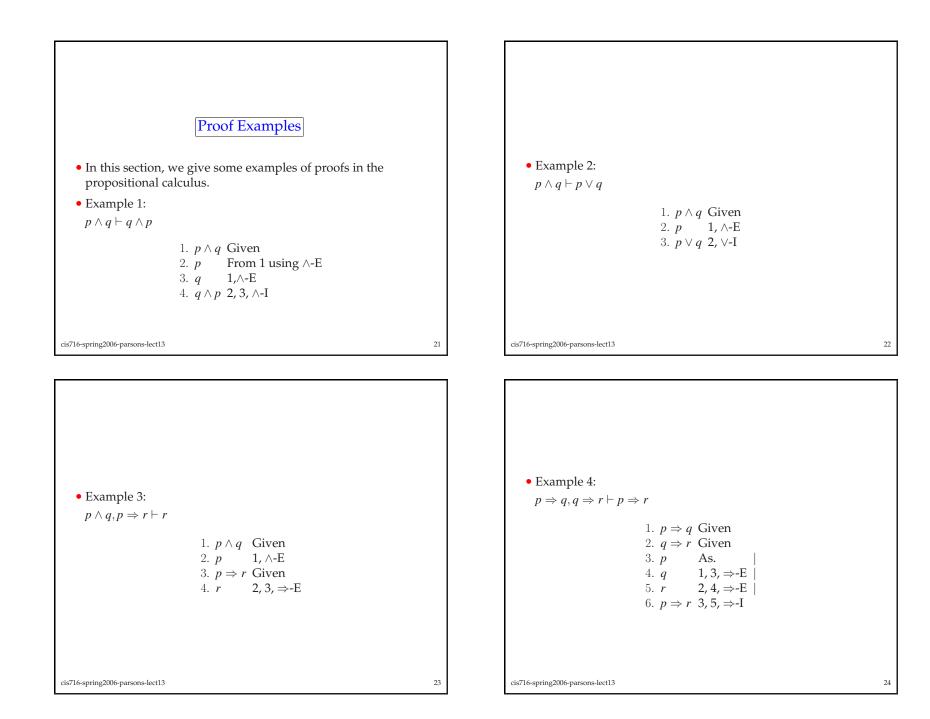
implies

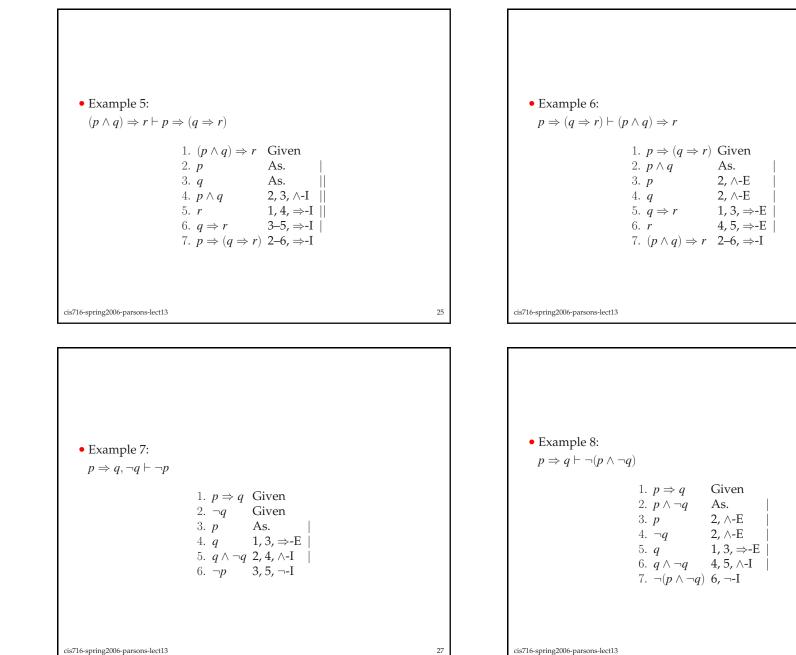
 $\phi_1,\ldots,\phi_n\vdash\phi.$ 

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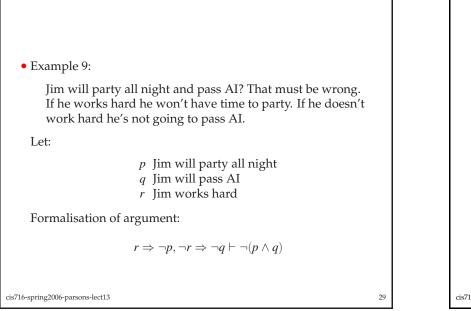
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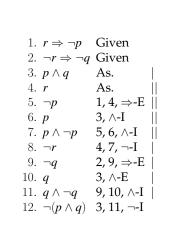












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Summary

- This lecture continued our look at propositional logic.
- It concentrated on proof theory, and gave examples of a number of different kinds of proof.
- Next lecture we will go on to look at predicate logic.

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