PREDICATE LOGIC

## First-Order Logic

- Aim of this lecture:
  to introduce *first-order predicate logic*.
- *More expressive* than propositional logic.
- Consider the following argument:
  - *all robots are ready*;
  - *X12 is a robot*;
  - therefore *X12 is ready*.
- Sense of this argument *cannot* be captured in propositional logic.
- Propositional logic is too *coarse grained* to allow us to represent and reason about this kind of statement.



## Syntax

- We shall now introduce a generalisation of propositional logic called first-order logic (FOL). This new logic affords us much greater expressive power.
- **Definition:** The alphabet of FOPL contains:
  1. a set of *constants*;
  2. a set of *variables*;
  3. a set of *function symbols*;
  4. a set of *predicates symbols*;
  5. the connectives $\vee$, $\neg$;
  6. the *quantifiers* $\forall$, $\exists$, $\exists_1$;
  7. the punctuation symbols ), (.



## Terms

- The basic components of FOL are called *terms*.
- Essentially, a term is an object that *denotes* some object other than $\top$ or $\bot$.
- The simplest kind of term is a *constant*.
- A value such as 8 is a constant.
- The *denotation* of this term is the number 8.
- Note that a constant and the value it denotes are different!
- Aliens don't write "8" for the number 8, and nor did the Romans.

- The second simplest kind of term is a *variable*.
- A variable can stand for anything in the *domain of discourse*.
- The domain of discourse (usually abbreviated to domain) is the set of all objects under consideration.
- Sometimes, we assume the set contains "everything".
- Sometimes, we explicitly *give* the set, and *state* what variables/constants can stand for.



## Functions

- We can now introduce a more complex class of terms — *functions*.
- The idea of functional terms in logic is similar to the idea of a function in programming.
- Recall that in programming, a function is a procedure that takes some arguments, and *returns a value*.

  In C:

```
T f(T1 a1, ..., Tn an)
```

  this function takes $n$ arguments; the first is of type `T1`, the second is of type `T2`, and so on. The function returns a value of type `T`.
- In FOL, we have a set of *function symbols*; each symbol corresponds to a particular function. (It denotes some function.)



- Each function symbol is associated with a number called its *arity*. This is just the number of arguments it takes.
- A *functional term* is built up by *applying* a function symbol to the appropriate number of terms.
- Formally ...

  **Definition:** Let $f$ be an arbitrary function symbol of arity $n$. Also, let $\tau_1, \ldots, \tau_n$ be terms. Then

$$f(\tau_1, \ldots, \tau_n)$$

  is a functional term.



- All this sounds complicated, but isn't. Consider a function *plus*, which takes just two arguments, each of which is a number, and returns the first number added to the second.

  Then:
  - *plus*$(2, 3)$ is an acceptable functional term;
  - *plus*$(0, 1)$ is acceptable;
  - *plus*$($*plus*$(1, 2), 4)$ is acceptable;
  - *plus*$($*plus*$($*plus*$(0, 1), 2), 4)$ is acceptable;

- In maths, we have many functions; the obvious ones are

$$+ \quad - \quad / \quad * \quad \sqrt{\quad} \quad \sin \quad \cos \quad \ldots$$

- The fact that we write

$$2 + 3$$

instead of something like

$$plus(2, 3)$$

is just convention, and is not relevant from the point of view of logic; all these are functions in exactly the way we have defined.



- Using functions, constants, and variables, we can build up *expressions*, e.g.:

$$(x + 3) * \sin 90$$

(which might just as well be written

$$times(plus(x, 3), sin(90))$$

for all it matters.)



## Predicates

- In addition to having terms, FOL has *relational operators*, which capture *relationships* between objects.
- The language of FOL contains *predicate symbols*.
- These symbols stand for *relationships between objects*.
- Each predicate symbol has an associated *arity* (number of arguments).
- **Definition:** Let $P$ be a predicate symbol of arity $n$, and $\tau_1, \ldots, \tau_n$ are terms.

  Then

$$P(\tau_1, \ldots, \tau_n)$$

is a predicate, which will either be $\top$ or $\bot$ under some interpretation.



- EXAMPLE. Let $gt$ be a predicate symbol with the intended interpretation 'greater than'. It takes two arguments, each of which is a natural number.

  Then:

  – $gt(4, 3)$ is a predicate, which evaluates to $\top$;
  – $gt(3, 4)$ is a predicate, which evaluates to $\bot$.

- The following are standard mathematical predicate symbols:

$$> \; < \; = \; \geq \; \leq \; \neq \; \ldots$$

- The fact that we are normally write $x > y$ instead of $gt(x, y)$ is just convention.

- We can build up more complex predicates using the connectives of propositional logic:

$$(2 > 3) \land (6 = 7) \lor (\sqrt{4} = 2)$$

- So a predicate just expresses a relationship between some values.
- What happens if a predicate contains *variables*: can we tell if it is true or false?

  Not usually; we need to know an *interpretation* for the variables.
- A predicate that contains no variables is a proposition.



- Predicates of arity 1 are called *properties*.
- EXAMPLE. The following are properties:

$$\begin{array}{l} \mathit{Woman}(x) \\ \mathit{Clever}(x) \\ \mathit{Powerful}(x). \end{array}$$

- We interpret $P(x)$ as saying $x$ is in the set $P$.
- Predicate that have arity 0 (i.e., take no arguments) are called *primitive propositions*.

  These are identical to the primitive propositions we saw in propositional logic.



## Quantifiers

- We now come to the central part of first order logic: *quantification*.
- Consider trying to represent the following statements:
  - *all people have a mother*;
  - *every positive integer has a prime factor*.
- We can't represent these using the apparatus we've got so far; we need *quantifiers*.



- We use three quantifers:

  $\forall$ — *the universal quantifier*;
  is read 'for all...'

  $\exists$ — *the existential quantifier*;
  is read 'there exists...'

  $\exists_1$ — *the unique quantifier*;
  is read 'there exists a unique...'

- The simplest form of quantified formula is as follows:

$$\textit{quantifier variable} \cdot \textit{predicate}$$

where

  - *quantifier* is one of $\forall$, $\exists$, $\exists_1$;
  - *variable* is a variable;
  - and *predicate* is a predicate.



## Examples

- $\forall x \cdot Person(x) \Rightarrow Mortal(x)$

  'For all $x$, if $x$ is a person, then $x$ is mortal.'

  (i.e. all people are mortal)

- $\forall x \cdot Person(x) \Rightarrow \exists_1 y \cdot Woman(y) \wedge MotherOf(x, y)$

  'For all $x$, if $x$ is a person, then there exists exactly one $y$ such that $y$ is a woman and the mother of $x$ is $y$.'

  (i.e., every person has exactly one mother).



- $\exists m \cdot Robot(r) \wedge RobotState(r, ready)$

  'There exists a robot that is in the ready state.'

- $\forall r \cdot Reactor(r) \Rightarrow \exists_1 t \cdot (100 \leq t \leq 1000) \wedge temp(r) = t$

  'Every reactor will have a temperature in the range 100 to 1000.'



- $\exists n \cdot posInt(n) \wedge n = (n * n)$

  'Some positive integer is equal to its own square.'

- $\exists c \cdot ECCountry(c) \wedge Borders(c, Albania)$

  'Some EC country borders Albania.'

- $\forall m, n \cdot Person(m) \wedge Person(n) \Rightarrow \neg Superior(m, n)$

  'No person is superior to another.'

- $\forall m \cdot Person(m) \Rightarrow \neg\exists n \cdot Person(n) \wedge Superior(m, n)$

  Ditto.

## Domains & Interpretations

- Suppose we have a formula $\forall x \cdot P(x)$.
  What does $x$ *range over*?
  Physical objects, numbers, people, times, ...?
- Depends on the *domain* that we intend.
- Often, we *name* a domain to make our intended interpretation clear.



- Suppose our intended interpretation is the +ve integers. Suppose $>, +, *, \ldots$ have the usual mathematical interpretation.
- Is this formula:

$$\exists n \cdot n = (n * n)$$

  *satisfiable* under this interpretation?
- Now suppose that our domain is all living people, and that $*$ means "is the child of".
- Is the formula satisfiable under this interpretation?



## Comments

- Note that universal quantification is similar to conjunction.
  Suppose the domain is the numbers $\{2, 4, 6\}$. Then

$$\forall n \cdot Even(n)$$

  is the same as

$$Even(2) \wedge Even(4) \wedge Even(6).$$

- Existential quantification is similar to *disjunction*. Thus with the same domain,

$$\exists n \cdot Even(n)$$

  is the same as

$$Even(2) \vee Even(4) \vee Even(6).$$



- The universal and existential quantifiers are in fact *duals* of each other:

$$\forall x \cdot P(x) \Leftrightarrow \neg\exists x \cdot \neg P(x)$$

  *Saying that everything has some property is the same as saying that there is nothing that does not have the property.*

$$\exists x \cdot P(x) \Leftrightarrow \neg\forall x \cdot \neg P(x)$$

  *Saying that there is something that has the property is the same as saying that its not the case that everything doesn't have the property.*

## Decidability

- In propositional logic, we saw that some formulae were tautologies — they had the property of being true under all interpretations.
- We also saw that there was a procedure which could be used to tell whether any formula was a tautology — this procedure was the truth-table method.
- A formula of FOL that is true under all interpretations is said to be *valid*.
- So in theory we could check for validity by writing down all the possible interpretations and looking to see whether the formula is true or not.



- Unfortuately in general we can't use this method.
- Consider the formula:

$$\forall n \cdot Even(n) \Rightarrow \neg Odd(n)$$

- There are an infinite number of interpretations.
- Is there any other procedure that we can use, that will be guaranteed to tell us, in a finite amount of time, whether a FOL formula is, or is not, valid?
- The answer is *no*.
- FOL is for this reason said to be *undecidable*.



## Proof in FOL

- Proof in FOL is similar to PL; we just need an extra set of rules, to deal with the quantifiers.
- FOL *inherits* all the rules of PL.
- To understand FOL proof rules, need to understand *substitution*.
- The most obvious rule, for $\forall$-E.

  Tells us that if everything in the domain has some property, then we can infer that any *particular* individual has the property.

$$\frac{\vdash \forall x \cdot \phi(x);}{\vdash \phi(a)} \; \forall\text{-E} \quad \text{for any } a \text{ in the domain}$$

  Going from *general* to *specific*.



- Example 1.

  Let's use $\forall$-E to get the Socrates example out of the way.

$$Person(s); \forall x \cdot Person(x) \Rightarrow Mortal(x) \vdash Mortal(s)$$

| | | |
|---|---|---|
| 1. | $Person(s)$ | Given |
| 2. | $\forall x \cdot Person(x) \Rightarrow Mortal(x)$ | Given |
| 3. | $Person(s) \Rightarrow Mortal(s)$ | 2, $\forall$-E |
| 4. | $Mortal(s)$ | 1, 3, $\Rightarrow$-E |

- We can also go from the general to the slightly less specific!

$$\frac{\vdash \forall x \cdot \phi(x);}{\vdash \exists x \cdot \phi(x)} \ \exists\text{-I(1)} \ \text{if domain not empty}$$

Note the *side condition*.

The $\exists$ quantifier *asserts the existence* of at least one object.

The $\forall$ quantifier does not.



- We can also go from the very specific to less specific.

$$\frac{\vdash \phi(a);}{\vdash \exists x \cdot \phi(x)} \ \exists\text{-I(2)}$$

- In other words once we have a concrete example, we can infer there exists something with the property of that example.



- We often informally make use of arguments along the lines...
  1. We know somebody is the murderer.
  2. Call this person $a$.
  3. ...

  (Here, $a$ is called a *Skolem constant*.)
- We have a rule which allows this, but we have to be careful how we use it!

$$\frac{\vdash \exists x \cdot \phi(x);}{\vdash \phi(a)} \ \exists\text{-E} \ a \text{ doesn't occur elsewhere}$$



- Here is an *invalid* use of this rule:

| | | |
|---|---|---|
| 1. | $\exists x \cdot Boring(x)$ | Given |
| 2. | $Lecture(AI)$ | Given |
| 3. | $Boring(AI)$ | 1, $\exists$-E |

- (The conclusion may be true, the argument isn't sound.)

- Another kind of reasoning:
  - Let $a$ be arbitrary object.
  - ... (some reasoning) ...
  - Therefore $a$ has property $\phi$
  - Since $a$ was arbitrary, it must be that every object has property $\phi$.
- Common in mathematics:

  Consider a positive integer $n$ ... so $n$ is either a prime number or divisible by a smaller prime number ... so every positive integer is either a prime number or divisible by a smaller prime number.



- If we are careful, we can also use this kind of reasoning:

$$\frac{\vdash \phi(a);}{\vdash \forall x \cdot \phi(x)} \forall\text{-I} \quad a \text{ is arbitrary}$$

- Invalid use of this rule:

| | | |
|---|---|---|
| 1. | $Boring(AI)$ | Given |
| 2. | $\forall x \cdot Boring(x)$ | 1, $\forall$-I |



- Example 2:
  1. Everybody is either happy or rich.
  2. Simon is not rich.
  3. Therefore, Simon is happy.

  Predicates:
  - $H(x)$ means $x$ is happy;
  - $R(x)$ means $x$ is rich.
- Formalisation:

$$\forall x.H(x) \vee R(x); \neg R(Simon) \vdash H(Simon)$$



| | | |
|---|---|---|
| 1. | $\forall x.H(x) \vee R(x)$ | Given |
| 2. | $\neg R(Simon)$ | Given |
| 3. | $H(Simon) \vee R(Simon)$ | 1, $\forall$-E |
| 4. | $\neg H(Simon) \Rightarrow R(Simon)$ | 3, defn $\Rightarrow$ |
| 5. | $\neg H(Simon)$ | As. \| |
| 6. | $R(Simon)$ | 4, 5, $\Rightarrow$-E \| |
| 7. | $R(Simon) \wedge \neg R(Simon)$ | 2, 6, $\wedge$-I \| |
| 8. | $\neg\neg H(Simon)$ | 5, 7, $\neg$-I |
| 9. | $H(Simon) \Leftrightarrow \neg\neg H(Simon)$ | PL axiom |
| 10. | $(H(Simon) \Rightarrow \neg\neg H(Simon)) \wedge (\neg\neg H(Simon) \Rightarrow H(Simon))$ | 9, defn $\Leftrightarrow$ |
| 11. | $\neg\neg H(Simon) \Rightarrow H(Simon)$ | 10,$\wedge$-E |
| 12. | $H(Simon)$ | 8, 11, $\Rightarrow$-E |

## Logic-Based Agents

- When we started talking about logic, it was as a means of representing knowledge.
- We wanted to represent knowledge in order to be able to build agents.
- We now know enough about logic to do that.
- We will now see how a *logic-based agent* can be designed to perform simple tasks.
- Assume each agent has a *database*, i.e., set of FOL-formulae.

  These represent information the agent has about environment.



- We'll write $\Delta$ for this database.
- Also assume agent has set of *rules*.

  We'll write $R$ for this set of rules.
- We write $\Delta \vdash_R \phi$ if the formula $\phi$ can be proved from the database $\Delta$ using only the rules $R$.
- How to program an agent:

  *Write the agent's rules $R$ so that it should do action $a$ whenever $\Delta \vdash_R Do(a)$.*

  Here, $Do$ is a predicate.
- Also assume $A$ is set of actions agent can perform.



- The agent's operation:

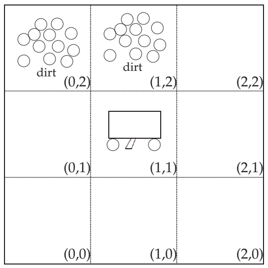
```
for each $a$ in $A$ do
    if $\Delta \vdash_R Do(a)$ then
        return $a$
    end-if
end-for
for each $a$ in $A$ do
    if $\Delta \nvdash_R \neg Do(a)$ then
        return $a$
    end-if
end-for
return $null$
```



- An example:

  We have a small robot that will clean up a house. The robot has sensor to tell it whether it is over any dirt, and a vacuum that can be used to suck up dirt. Robot always has an orientation (one of $n$, $s$, $e$, or $w$). Robot can move forward one "step" or turn right $90°$. The agent moves around a room, which is divided grid-like into a number of equally sized squares. Assume that the room is a $3 \times 3$ grid, and agent starts in square $(0, 0)$ facing north.

- Illustrated:






- Three *domain predicates* in this exercise:

| | |
|---|---|
| $\mathit{In}(x, y)$ | agent is at $(x, y)$ |
| $\mathit{Dirt}(x, y)$ | there is dirt at $(x, y)$ |
| $\mathit{Facing}(d)$ | the agent is facing direction $d$ |

- For convenience, we write rules as:

$$\phi(\ldots) \longrightarrow \psi(\ldots)$$

- First rule deals with the basic cleaning action of the agent

$$\mathit{In}(x, y) \wedge \mathit{Dirt}(x, y) \longrightarrow \mathit{Do}(\mathit{suck}) \tag{1}$$

- Hardwire the basic navigation algorithm, so that the robot will always move from $(0, 0)$ to $(0, 1)$ to $(0, 2)$ then to $(1, 2)$, $(1, 1)$ and so on.



- Once agent reaches $(2, 2)$, it must head back to $(0, 0)$.

$$\mathit{In}(0, 0) \wedge \mathit{Facing}(\mathit{north}) \wedge \neg \mathit{Dirt}(0, 0) \longrightarrow \mathit{Do}(\mathit{forward}) \tag{2}$$
$$\mathit{In}(0, 1) \wedge \mathit{Facing}(\mathit{north}) \wedge \neg \mathit{Dirt}(0, 1) \longrightarrow \mathit{Do}(\mathit{forward}) \tag{3}$$
$$\mathit{In}(0, 2) \wedge \mathit{Facing}(\mathit{north}) \wedge \neg \mathit{Dirt}(0, 2) \longrightarrow \mathit{Do}(\mathit{turn}) \tag{4}$$
$$\mathit{In}(0, 2) \wedge \mathit{Facing}(\mathit{east}) \longrightarrow \mathit{Do}(\mathit{forward}) \tag{5}$$

- Other considerations:
  - *adding* new information after each move/action;
  - *removing* old information.
- Suppose we scale up to $10 \times 10$ grid?



## Summary

- This lecture looked at predicate (or first order) logic.
- Predicate logic is a generalisation of propositional logic.
- The generalisation requires the use of quantifiers, and these need special rules for handling them when doing inference.
- We looked at how the proof rules for propositional logic need to be extended to handle quantifiers.
- Finally, we looked at how logic might be used to control an agent.