# KINEMATICS II

## What went before

- Last time we looked at two parts of the mathematics of robot motion.
- Overall motion, making simplifying assumptions
  - Robot as a body moving in a plane.
  - Forward and reverse kinematic models.
- Motion of individual wheels.
  - Equations relating to wheel spin and slideways motion.
  - Fixed and steerable standard wheels have constraints.
  - Swedish and spherical wheels, as well as castors, do not have constraints.

# Today

- Take individual constraints on wheels, and use these to establish constraints on the robot as a whole.
- Gives us a more accurate kinematic picture of the whole robot.
  - Robot as a constrained body moving in a plane.
- Tells us how the design of the robot constrains its ability to move.
- Gives us precise notions of:
  - Mobility
  - Steerability
  - Maneuverability

## Kinematic constraints

- How does the design of a robot with $M$ wheels constrain how the robot moves?
  - How does a differential drive robot move compared with a bicycle?
- Five categories of wheel:
  - Fixed standard
  - Steerable standard
  - Castors
  - Swedish
  - Spherical
- Only fixed and steerable standard wheels have any constraints.

- A fixed standard wheel, radius $r$, polar coordinates $l$ and $\alpha$ to some reference point on the chassis, wheel at $\beta$ to chassis, chassis moving at $\dot{\xi}_I$ has a rolling constraint:

$$[\sin(\alpha+\beta) \; -\cos(\alpha+\beta) \; (-l)\cos\beta]\, R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

where $\dot{\varphi}$ is the rate of rotation of the wheel about its axle and:

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

translates from the global frame of reference to the local frame.

- This says that the wheel does not slip at its point of contact with the ground.

- Similarly, we have a sliding constraint:

$$[\cos(\alpha+\beta)\,\sin(\alpha+\beta)\boldsymbol{l}\sin\beta]\,\boldsymbol{R}(\theta)\dot{\xi}_I = 0$$

- This says that the wheel does not move perpendicular to its plane of rotation.
- We can write identical expressions for a steerable standard wheel.

## From constraints on wheels to constraints on robots

- Consider we have a robot with $N$ wheels.
  - $N_f$ are fixed standard wheels
  - $N_s$ are steerable standard wheels
- Deal with this by considering fixed wheels all together, and steerable wheels all together.
- $\beta_s(t)$ is a vector that describes the steering angles of all $N_s$ steerable wheels.
- $\beta_f$ is a vector that describes the orientation of all $N_f$ fixed wheels.

- $\varphi_f(t)$ is a vector that describes the rotation of the fixed wheels.
- $\varphi_s(t)$ is a vector that describes the rotation of the steerable wheels.
- $\varphi(t)$ collects these together:

$$\varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

- We can now write the rolling constraints of all the wheels in a way analogous to the constraint for one wheel:

$$J_1(\beta_s)R(\theta)\dot{\xi}_I - J_2\dot{\varphi} = 0$$

where $J_2$ gives wheel radii, and $J_1$ relates wheels to the motion along their planes:

$$J_2 = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_{N_f+N_s} \end{bmatrix}$$

$$J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}$$

- $J_{1f}$ and $J_{1s}(\beta_s)$ are relate wheels to motion for fixed and steerable wheels respectively.

- We can do the same thing for sliding constraints, giving:

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0$$

where

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

- These two expressions then summarize all the constraints on the root due to its wheels.
- The sliding constraint (the second one) has the biggest impact on what a robot can do.

# Maneuverability of a mobile robot

- The mobility of a chassis is its ability to directly move in the environment.
- An unconstrained robot can move in any direction at any time, and therefore follow any path through the environment.
- There are two parts to mobility:
  - Instantaneous motion
  - Steering

  which is (roughly speaking) the difference between omni-drive and a car chassis.
- The overall maneuverability of a chassis is a combination of freedom from sliding constraints and steerability.

- Sliding constraints require that:

$$C_{1f}R(\theta)\dot{\xi}_I = 0$$

and

$$C_{1s}(\beta_s)R(\theta)\dot{\xi}_I = 0$$

- These put contraints on $R(\theta)\dot{\xi}_I$, and thus on the motion that the robot can perform.
- One view:

  $R(\theta)\dot{\xi}_I$ must belong to the null space of $C_1(\beta_s)$

  meaning that motion is in some space $N$ such that for any $n \in N$

$$C_1(\beta_s)n = 0$$

- Another view: instantaneous center of rotation.

## Instantaneous center of rotation

## Back to maneuverability

- The range of possible motion is determined by the set of *independent* constraints.
- Related to the *rank* of $C_1(\beta_s)$.
- More constraints = greater rank of $C_1(\beta_s)$
- More constraints = less flexibility in mobility of the robot.

- A robot with one fixed standard wheel.
  - One constraint.
  - $C_1(\beta_s)$ has rank one.
- A robot with two fixed standard wheels in differential drive:
  - Two constraints, but not independent:
  - $C_1(\beta_s)$ has rank one.
- A robot with two fixed standard wheels in bicycle drive:
  - Two constraints.
  - $C_1(\beta_s)$ has rank two

- Range of possible values for *rank* $[C_1(\beta_s)]$:

$$0 \leq C_1(\beta_s) \leq 3$$

- $C_1(\beta_s) = 0$. No constraints, and no standard wheels.
- $C_1(\beta_s) = 3$. Fully constrained since only three dimensions to constrain.
- $C_1(\beta_s) = 3$. No motion possible.

## Degree of mobility

- The null space $N$ of $C_1(\beta_s)$ defines how the robot can move just by changing wheel velocity.
- The dimensionality of $N$ measures the degrees of freedom under the robot's control just by altering velocity.
- Define *degree of mobility* $\delta_m$:

$$\begin{aligned}\delta_m &= dim\, N\left[C_1(\beta_s)\right] \\ &= 3 - rank\left[C_1(\beta_s)\right]\end{aligned}$$

- Differential drive robot:
  - $\delta_m = 2$
  - Robot can change both orientation and position on its current path just by changing wheel speed.
- Bicycle drive robot:
  - $\delta_m = 1$
  - Robot can only change position on its current path by changing wheel speed.
  - Needs steerable wheel to change orientation.

## Degree of steerability

- Degree of mobility captures what can be done just with changes in wheel velocity.
- What about steerable wheels?
- They don't have an instantaneous effect, but they do have an effect over time.
- Degree of steerability $\delta_s$

$$\delta_s = rank\left[C_{1s}(\beta_s)\right]$$

  $C_{1s}(\beta_s)$ is the "steerable" bit of $C_1(\beta_s)$.

- The bigger the rank of $C_{1s}(\beta_s)$, the more steerable the robot.

- $0 \leq \delta_s \leq 2$
- $\delta_s = 0$ implies no steerable wheels.
- $\delta_s = 1$ implies one independent steerable wheel.
  - As in a car where two steerable wheels share one axle
- $\delta_s = 2$ only possible if no standard wheels
  - “Two-steer”

# Degree of manueverability

- The *degree of maneuverability* depends on both mobility and steerability:

$$\delta_M = \delta_m + \delta_s$$

- Maneuverability includes both the degrees of freedom that can be manipulated instantaneously through changes in wheel velocity, and those that can be manipulated through steering.
- As a result, robots with the same $\delta_M$ are notnecessarily equivalent.
- We can see that by looking at different wheel configurations.

## On to workspace

- What is important is how the robot can move in its environment.
- Degrees of freedom (DOF)
  - Feature of the workspace
- Differentiable degrees of freedom
  - Feature of the robot
  - Number of independently achievable velocities
  - Equal to degree of mobility $\delta_m$.

## Summary

- These notes have discussed robot kinematics.
- In particular, they have considered how we can develop constraints on a whole robot from constraints on a set of wheels.
- Furthermore, they have discussed how this set of constraints on a robot can be used to develop notions of *mobility*, *steerability*, and *maneuverability*.