# UNCERTAINTY AND DECISION MAKING

Introduction

- This week we'll finish off talking about handling uncertainty
  - Probability
- We'll then go on to talk about how we make decisions under uncertainty.
- Requires us to use probability to handle the uncertainty.
- Also requires us to find a way to rate outcomes
  - Utility
- We will pick up where we finished last time with computing probability values in a Bayesian network.

Inference by enumeration

- Simplest approach to evaluating the network is to do just as we did for the dentist example.

- Difference is that we use the structure of the network to tell us which sets of joint probabilities to use.

    – Thanks Professor Markov

- Gives us a slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation
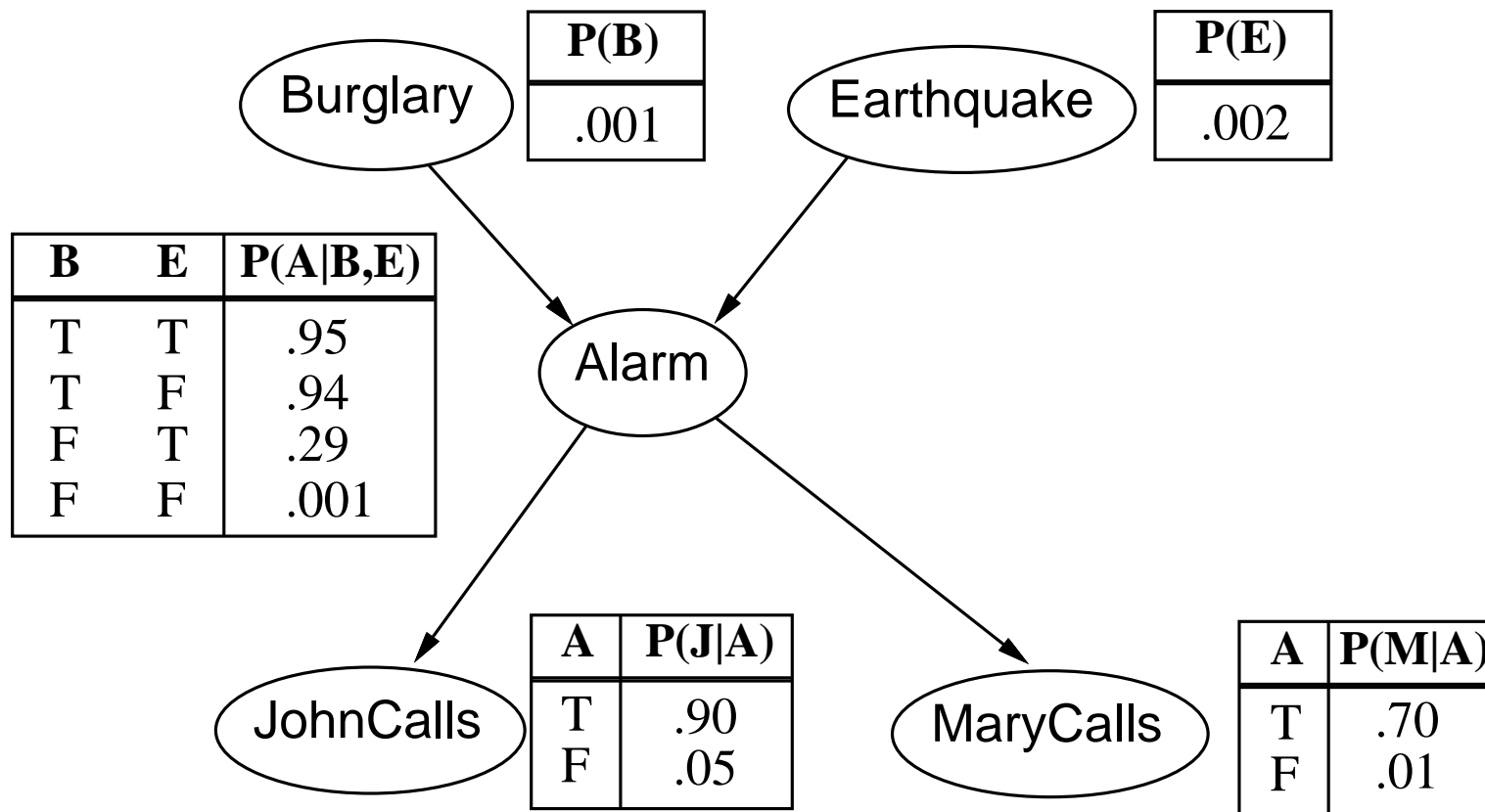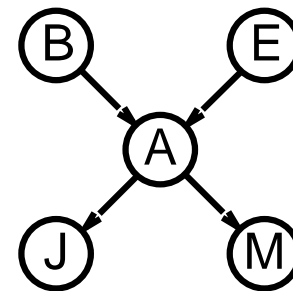
- Simple query on the burglary network.

$$\mathbf{P}(B|j,m)$$
$$= \mathbf{P}(B,j,m)/P(j,m)$$
$$= \alpha \mathbf{P}(B,j,m)$$
$$= \alpha \ \Sigma_e \ \Sigma_a \ \mathbf{P}(B,e,a,j,m)$$

- Rewrite full joint entries using product of CPT entries:

$$\mathbf{P}(B|j,m)$$
$$= \alpha \ \Sigma_e \ \Sigma_a \ \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$$
$$= \alpha\mathbf{P}(B) \ \Sigma_e \ P(e) \ \Sigma_a \ \mathbf{P}(a|B,e)P(j|a)P(m|a)$$

Enumeration algorithm

function ENUMERATION-ASK($X, \mathbf{e}, bn$) **returns** a distribution over X

    inputs: X, the query variable

            $\mathbf{e}$, observed values for variables $\exists$

            bn, a Bayesian network with variables $\{X\} \cup \exists \cup \mathbf{Y}$

    $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
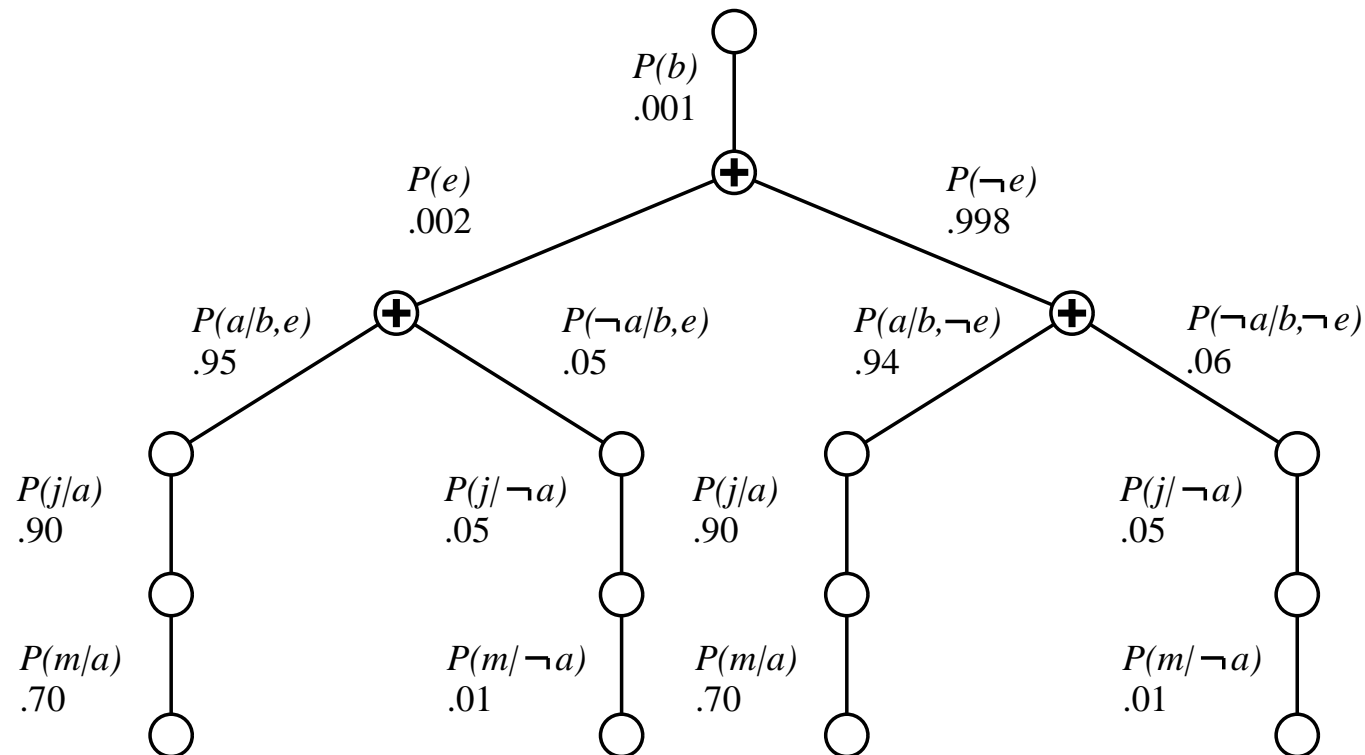
    for each value x_i of X **do**

        extend $\mathbf{e}$ with value x_i for X

        $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(VARS[bn], $\mathbf{e}$)

    return NORMALIZE($\mathbf{Q}(X)$)

function ENUMERATE-ALL(*vars*, **e**) **returns** a real number
   if EMPTY?(*vars*) **then return** 1.0
   $Y \leftarrow$ FIRST(*vars*)
   if Y has value y in **e**
      then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST(*vars*), **e**)
      else return $\Sigma_y \ P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST(*vars*),
e$_y$)

   where **e**$_y$ is **e** extended with $Y = y$

Evaluation tree

- Enumeration is inefficient: repeated computation
  - Computes $P(j|a)P(m|a)$ for each value of e

Complexity of exact inference

- *Singly connected* networks (or *polytrees*)

    - any two nodes are connected by at most one (undirected) path
    - time and space cost of variable elimination are $O(d^k n)$

- *Multiply connected* networks:

    - can reduce 3SAT to exact inference $\Rightarrow$ NP-hard
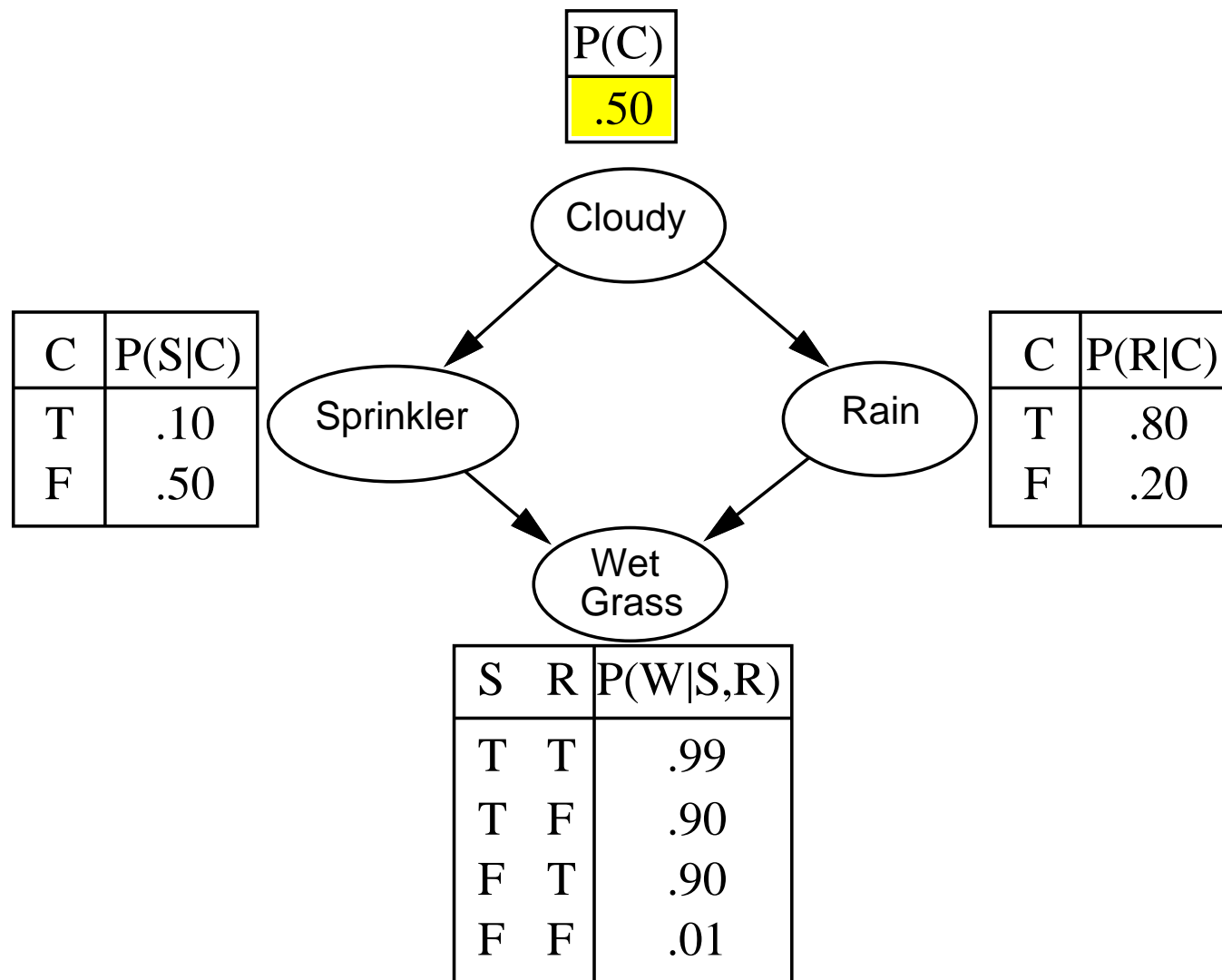   - equivalent to *counting* 3SAT models $\Rightarrow$ #P-complete

1. A v B v C

2. C v D v ¬A
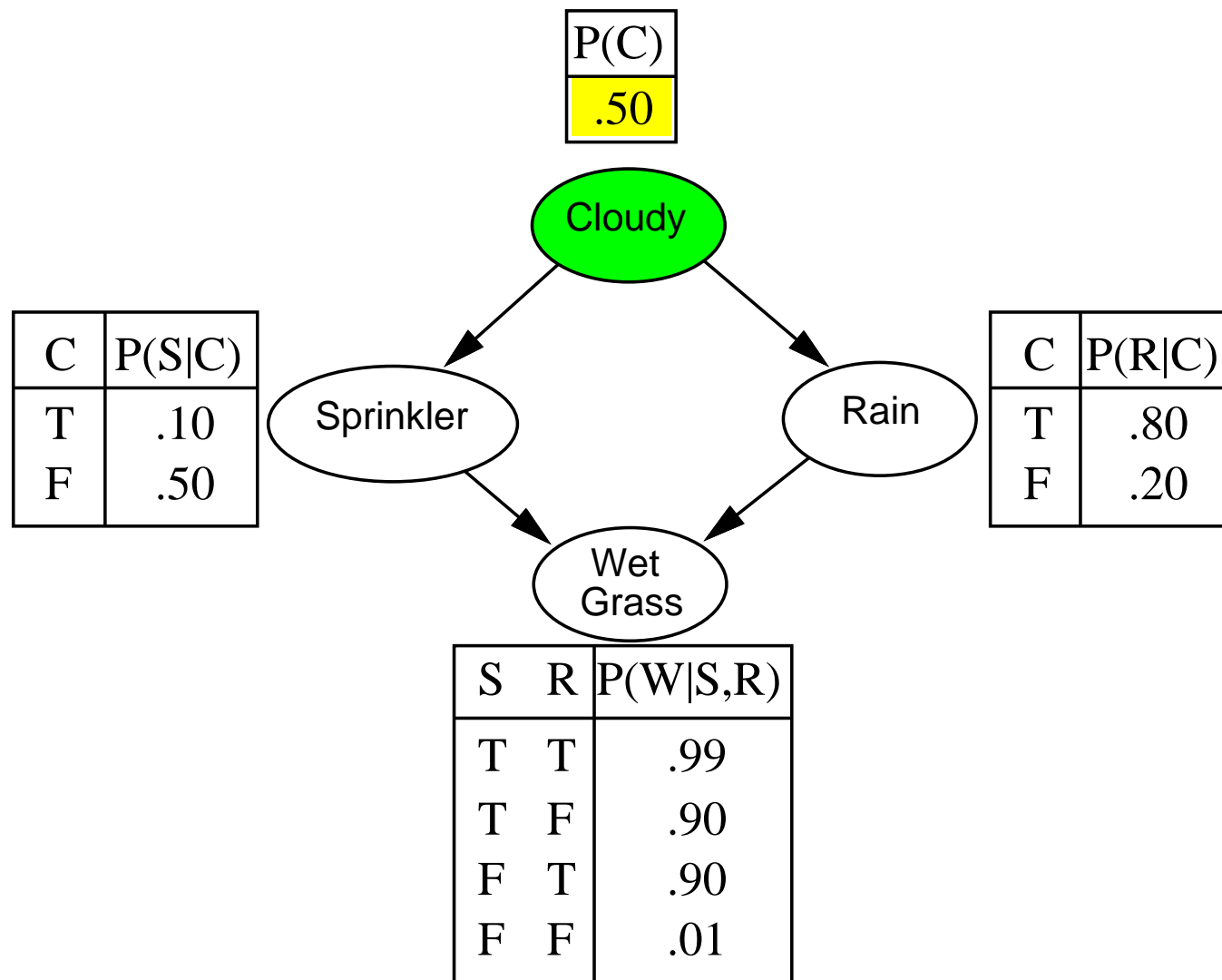
3. B v C v ¬D
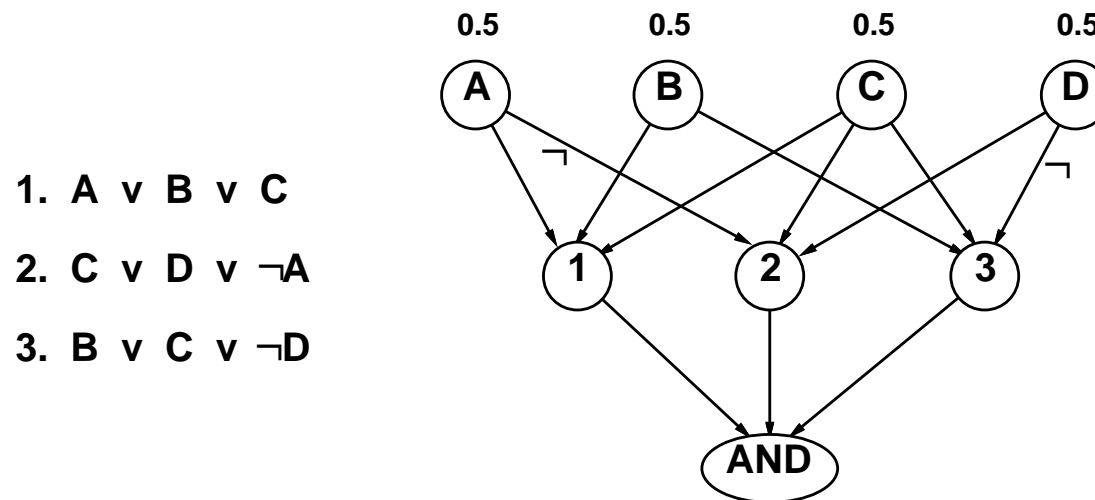
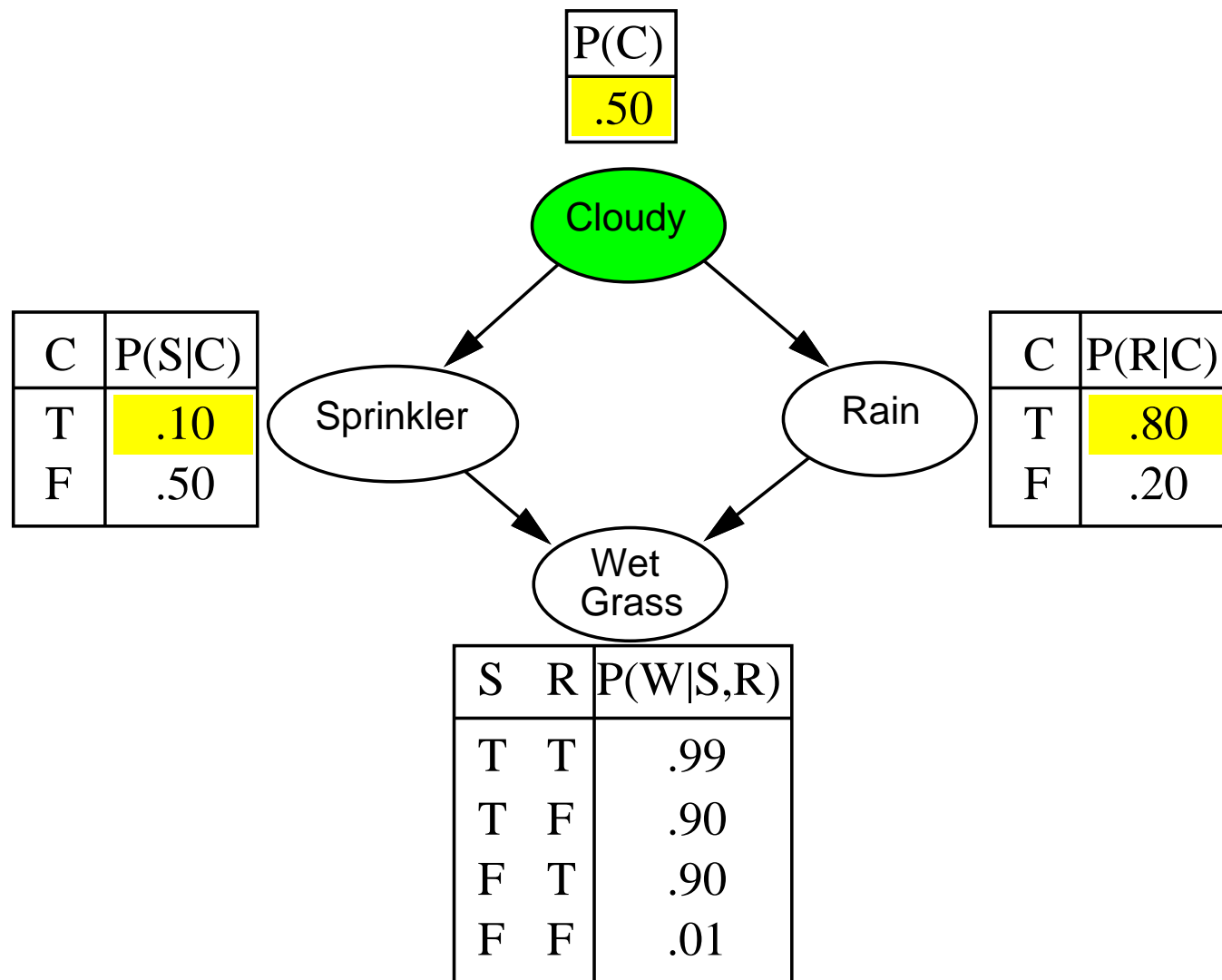# Inference by stochastic simulation

- Basic idea:

  1. Draw N samples from a sampling distribution S
  2. Compute an approximate posterior probability $\hat{P}$
  3. Show this converges to the true probability P

P(C)
.50

Cloudy

C	P(S|C)
T	.10
F	.50

Sprinkler

Rain

C	P(R|C)
T	.80
F	.20

Wet Grass

S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

P(C)

.50

Cloudy

C	P(S|C)
T	.10
F	.50

Sprinkler

Rain

C	P(R|C)
T	.80
F	.20

Wet
Grass

S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

P(C)

	.50

Cloudy

C	P(S|C)
T	.10
F	.50

Sprinkler

Rain

C	P(R|C)
T	.80
F	.20

Wet Grass

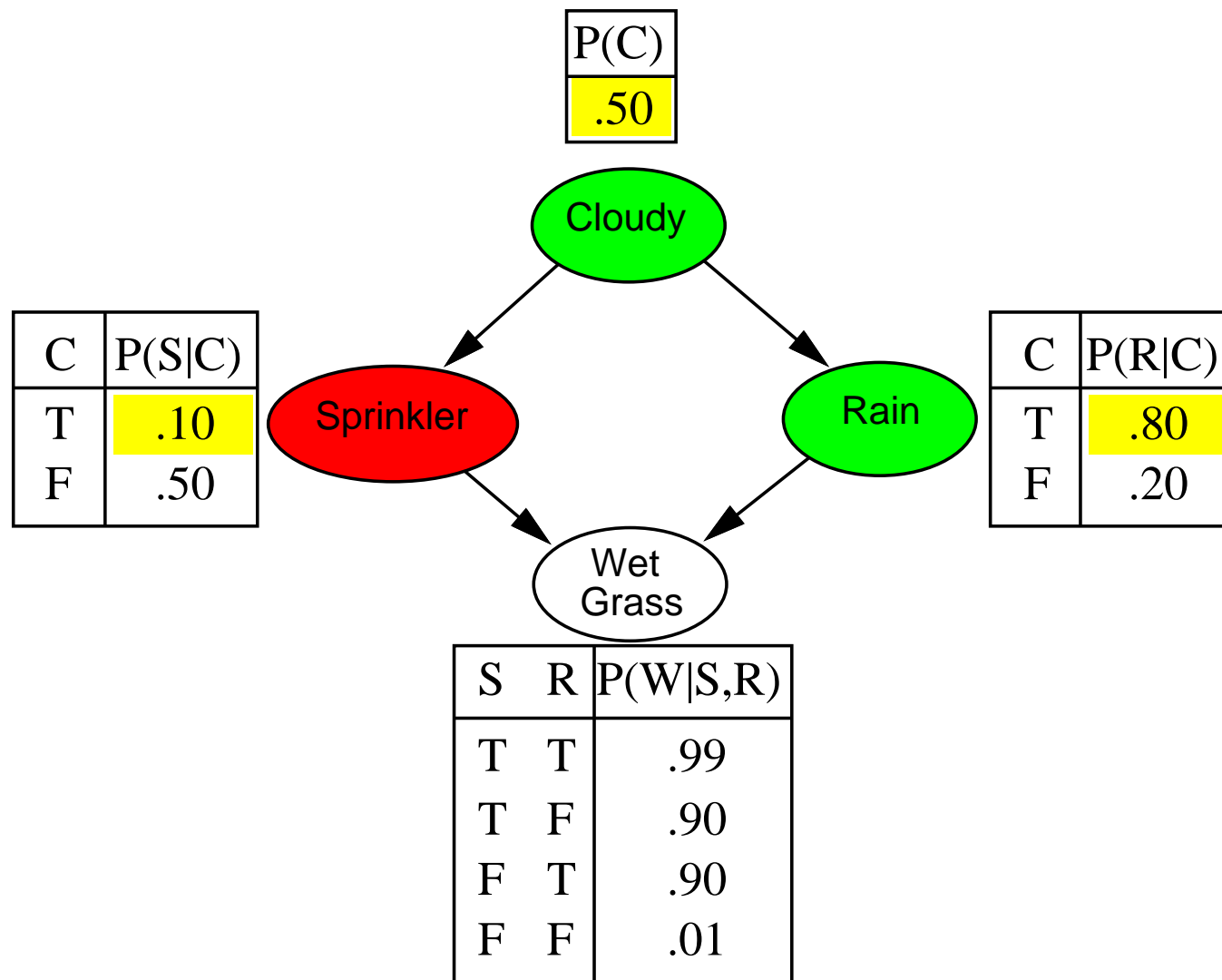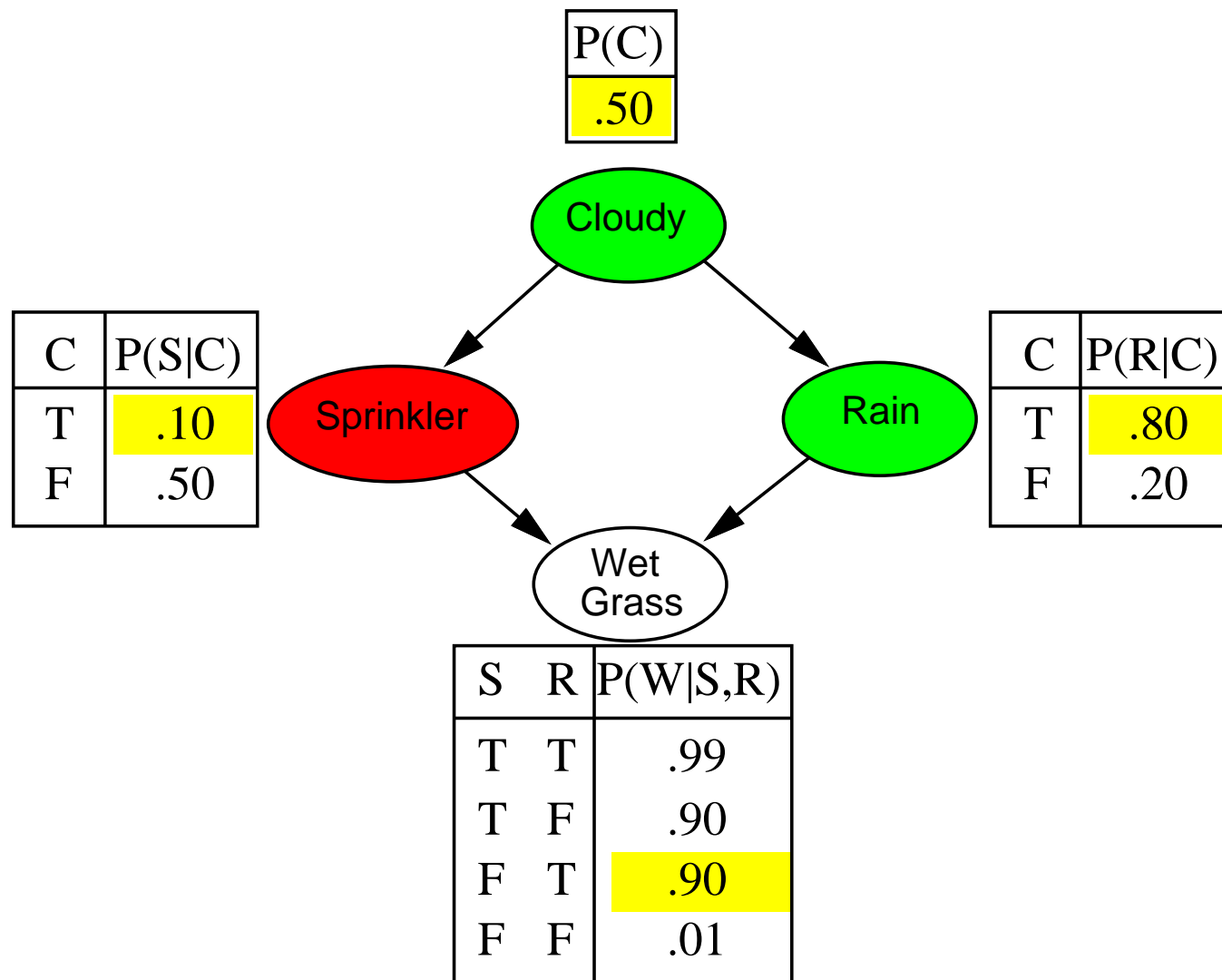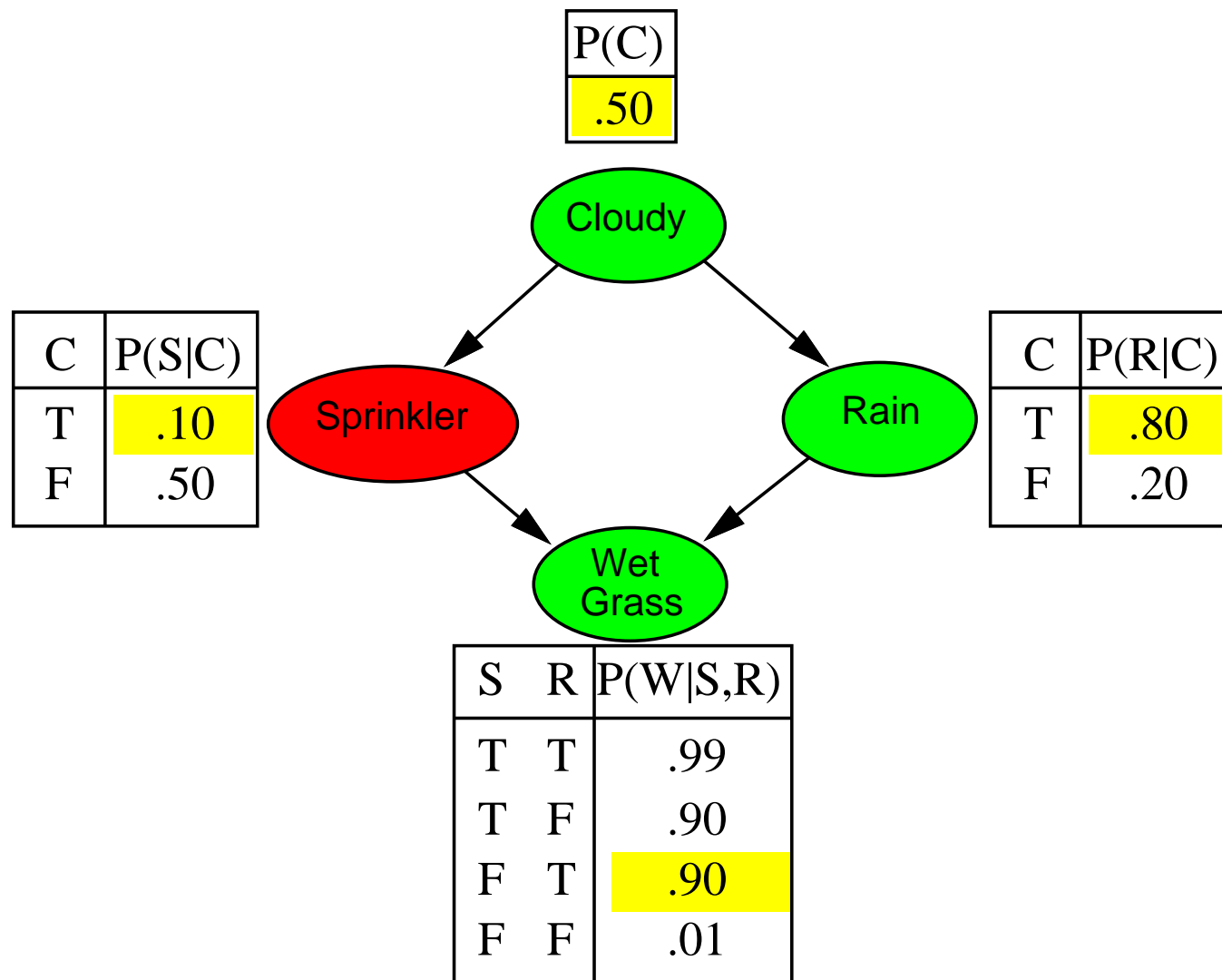| S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01
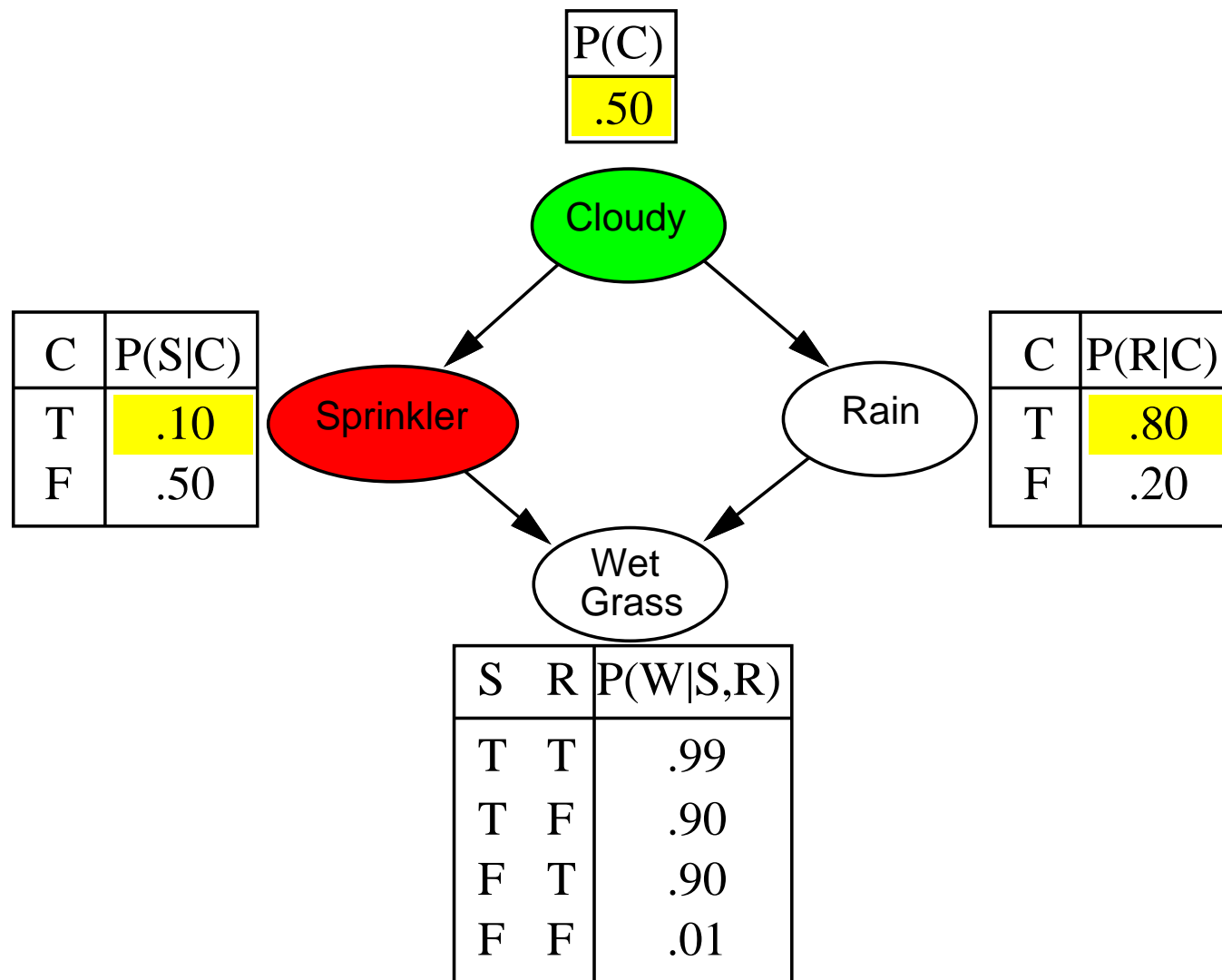
- So, this time we get the event

$$[true, false, true, true]$$

- If we repeat the process many times, we can count the number of times $[true, false, true, true]$ is the result.

- The proportion of this to the total number of runs is:

$$\mathbf{P}(c, \neg s, r, w)$$

- The more runs, the more accurate the probability.

- This algorithm:

  function PRIOR-SAMPLE(*bn*) **returns** an event sampled from *bn*
      inputs: *bn*, a belief network specifying joint distribution
  $\mathbf{P}(X_1, \ldots, X_n)$

      $\mathbf{x} \leftarrow$ an event with n elements
      for $i = 1$ **to** n **do**
          $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
              given the values of *Parents*(X_i) in $\mathbf{x}$
      return $\mathbf{x}$

  captures the *no evidence* case, which is what we just looked at.

- To get values with evidence, we need conditional probabilities

$$\mathbf{P}(X|\mathbf{e})$$

- Could just compute the joint probability and sum out the conditionals but that is inefficient.

- Better is to use *rejection sampling*

  - Sample from the network but reject samples that don't match the evidence.
  - If we want $\mathbf{P}(w|c)$ and our sample picks $\neg c$, we stop that run immediately.
  - For unlikely events, may have to wait a long time to get enough matching samples.

- Still inefficient.

• Likelihood weighting:

  – Version of *importance sampling*.

  – Fix evidence variable to *true*, so just sample relevant events.

  – Have to weight them with the likelihood that they fit the evidence.

  – Use the probabilities we know to weight the samples.

From probability to decision making

- What we have covered allows us to compute probabilities of interesting events.

- But *beliefs* alone are not so interesting to us.

- In the WW don't care so much if there is a pit in (2, 2), so much as we care whether we should go left or right.

- This is complicated because the world is uncertain.

  - Don't know the outcome of actions.
  - Non-deteriministic as well as partially observable

DA MAYOR: Mookie.

MOOKIE: Gotta go.

DA MAYOR: C'mere, Doctor.

DA MAYOR: Doctor, this is Da Mayor talkin'.

MOOKIE: OK. OK.

DA MAYOR: Doctor, always try to do the right thing.

MOOKIE: That's it?

DA MAYOR: That's it.

MOOKIE: I got it.

(Spike Lee, Do the Right Thing)

• I offer you the chance to take part in this gamble:

  – $0 one time in one hundred;
  – $1 89 times in one hundred;
  – $5 10 times in one hundred.

• Would you prefer this to $1.00?

- I offer you the chance to take part in this gamble:

    – $0 one time in one hundred;
    – $1 89 times in one hundred;
    – $5 10 times in one hundred.

- Would you prefer this to $1.50?

• I offer you the chance to take part in this gamble:

   – $0 one time in one hundred;
   – $1 89 times in one hundred;
   – $5 10 times in one hundred.

• Would you prefer this to $1.20?

• I offer you the chance to take part in this gamble:

  – $0 one time in one hundred;
  – $1 89 times in one hundred;
  – $5 10 times in one hundred.

• Would you prefer this to $1.40?

- We can't make this choice without thinking about how likely outcomes are.

- Although the first option is attractive, it isn't necessarily the best course of action (especially if the choice is iterated).

- Decision theory gives us a way of analysing this kind of situation.

- Consider being offered a bet in which you pay $2 if an odd number is rolled on a die, and win $3 if an even number appears.

- To analyse this prospect we need a *random variable X*, as the function:

$$X : \Omega \mapsto \Re$$

from the sample space to the values of the outcomes. Thus for $\omega \in \Omega$:

$$X(\omega) = \begin{cases} 3, & \text{if } \omega = 2, 4, 6 \\ -2, & \text{if } \omega = 1, 3, 5 \end{cases}$$

- The probability that X takes the value 3 is:

$$\Pr(\{2, 4, 6\}) = \Pr(\{2\}) + \Pr(\{4\}) + \Pr(\{6\})$$
$$= 0.5$$

- How do we analyse how much this bet is worth to us?

- To do this, we need to calculate the *expected value* of X.

- This is defined by:

$$E(X) = \sum_k k \Pr(X = k)$$

  where the summation is over all values of k for which $\Pr(X = k) \neq 0$.

- Here the expected value is:

$$E(X) = 0.5 \times 3 + 0.5 \times -2$$

- Thus the expected value of X is \$0.5, and we take this to be the value of the bet.

  – Not the value you will get.

- What is the expected value of this event:

    - $0 one time in one hundred;
    - $1 89 times in one hundred;
    - $5 10 times in one hundred.

- Would you prefer this to $1?

- And now we can make a first stab at defining what rational action is.

- Rational action is the choice of actions with the greatest expected value for the agent in question.

- The problem is then to decide what "value" is.

Decision theory

- One obvious way to define "value" is in terms of money.

- This has obvious applications in writing programs to trade stocks, or programs to play poker.

- The problem is that the value of a given amount of money to an individual is highly subjective.

- In addition, using monetary values does not take into account an individual's attitude to risk.

• As an example, consider a transaction which offered the following payoffs:

    – $0 one time in one hundred;

    – $1 million 89 times in one hundred;

    – $5 million 10 times in one hundred.

• Would you prefer this to a guaranteed $1 million?

- Utilities are a means of solving the problems with monetary values.

- Utilities are built up from preferences, and preferences are captured by a preference relation $\preceq$ which satisfies:

$$a \preceq b \quad \text{or} \quad b \preceq a$$
$$a \preceq b \ \text{and} \ b \preceq c \Rightarrow a \preceq c$$

- You have to be able to state a preference.

- Preferences are transitive.

- A function:

$$u : \Omega \mapsto \Re$$

is a utility function representing a preference relation $\preceq$ if and only if:

$$u(a) \leq u(b) \ \leftrightarrow \ a \preceq b$$

- With additional assumptions on the preference relation (to do with preferences between lotteries) Von Neumann and Morgenstern identified a sub-class of utility functions.

- These "Von Neumann and Morgenstern utility functions"are such that calculating expected utility, and choosing the action with the maximum expected utility is the "best" choice according to the preference relation.

- This is "best" in the sense that any other choice would disagree with the preference order.

- This is why the *maximum expected utility* decision criterion is said to be rational.

- To relate this back to the problem of an agent making a rational choice, consider an agent with a set of possible actions A available to it.

- Each $a \in A$ has a sample space Ω_a associated with it, and a set of possible outcomes s_a where $s_a \subseteq S_a$ and $S_a = 2^{\Omega_a}$.

- (This is a simplification since each s_a will usually be conditional on the state of the environment the agent is in.)

• The action a^* which a rational agent should choose is that which maximises the agent's utility.

• In other words the agent should pick:

$$a^* = arg \ \max_{a \in A} u(s_a)$$

• The problem is that in any realistic situation, we don't know which s_a will result from a given a, so we don't know the utility of a given action.

• Instead we have to calculate the expected utility of each action and make the choice on the basis of that.

- In other words, for the set of outcomes s_a of each action each a, the agent should calculate:
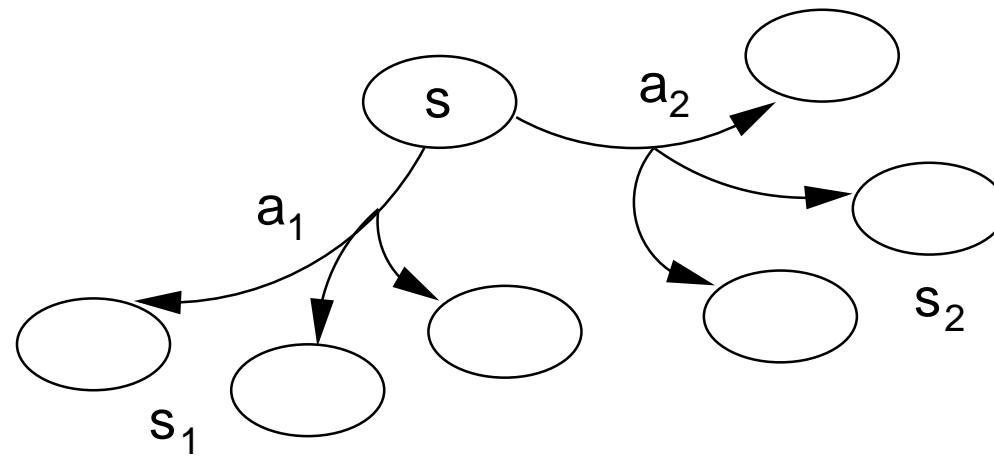
$$E(u(s_a)) = \sum_{s' \in s_a} u(s'). \Pr(s_a = s')$$

and pick the best.
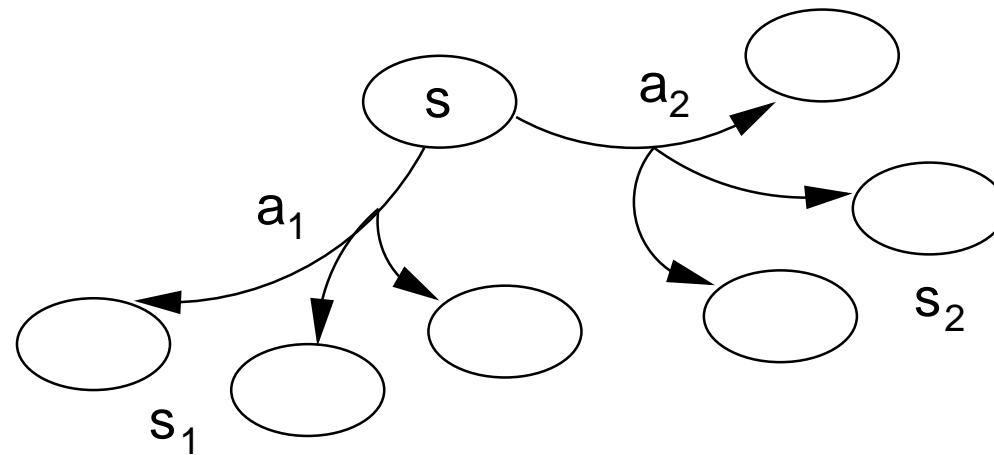
- Thus to be rational, an agent needs to choose a^* such that:

$$a^* = arg \max_{a \in A} \sum_{s' \in s_a} u(s'). \Pr(s_a = s')$$

- As an example, consider an agent which has to choose between tossing a coin, rolling a die, or receiving a payoff of $ 1.

- If the coin is chosen, then the agent gets $1.50 a head and $0.5 for a tail.

- If the die is chosen, the agent gets $5 if a six is rolled, $1 if a two or three is rolled, and nothing otherwise.

- What is the rational choice, assuming that the agent's preferences are (for once) modelled by monetary value?

- Well, we need to calculate the expected outcome of each choice.

- For doing nothing, we have a_1 = "receive payoff", s_{a_1} = {"get $1"}, $u($"get $1) = 1$ and $\Pr(s_{a_1} =$ "get $1) = 1$.

- Thus:

$$E(u(s_{a_1})) = 1$$

- If the coin is chosen, we have a_2 = "coin", s_{a_2} = {head, tail},

$$
\begin{aligned}
u(\text{head}) &= \$1.50 \\
u(\text{tail}) &= \$0.5
\end{aligned}
$$

and

$$
\begin{aligned}
\Pr(s_{a_2} = \text{head}) &= 0.5 \\
\Pr(s_{a_2} = \text{tail}) &= 0.5
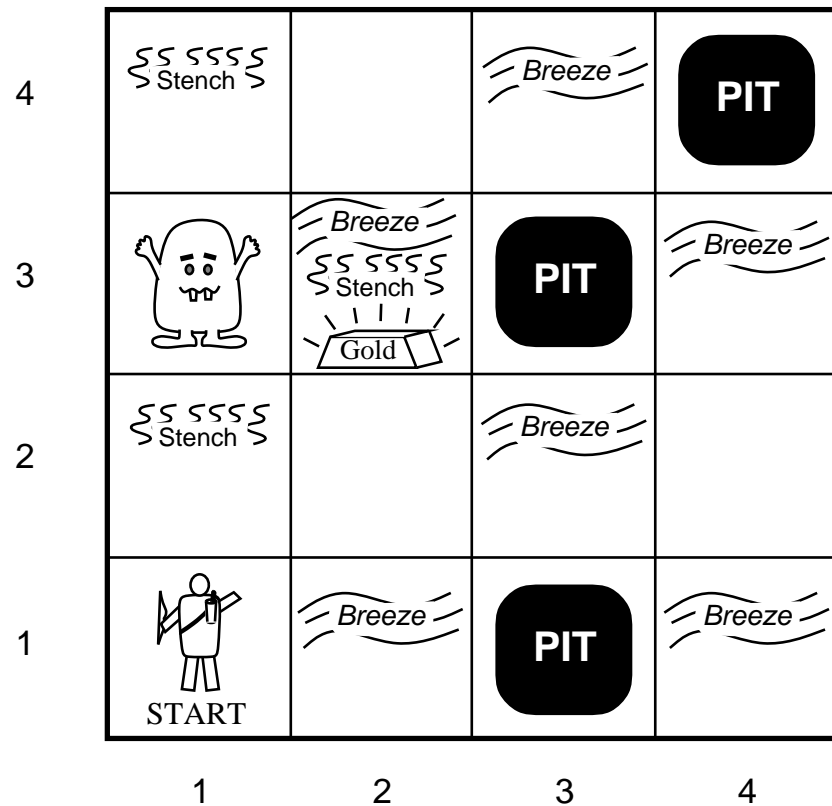\end{aligned}
$$

- Thus the expected utility is:

$$
\begin{aligned}
E(u(s_{a_2})) &= 0.5 \times 1.5 + 0.5 \times 0.5 \\
&= 1
\end{aligned}
$$

- Action a_3, rolling the die, can be analysed in a similar way, giving:

$$E(u(s_{a_3})) = 1.17$$

- Choosing to roll the die is the rational choice.

Decisions in the WW

- Actions have a range of outcomes.

- Forward has some probability of moving sideways

  – Not so silly with a robot

- Probabilities across action outcomes.

  – Given an action, probability of getting to some states

- Utilities for states.

- Given what we know about Bayesian networks, we can clearly deal with complex situations as far as probability is concerned.
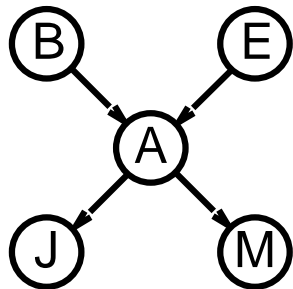
- Should I go home given that John calls and Mary doesn't?

- But what about more complex decisions?
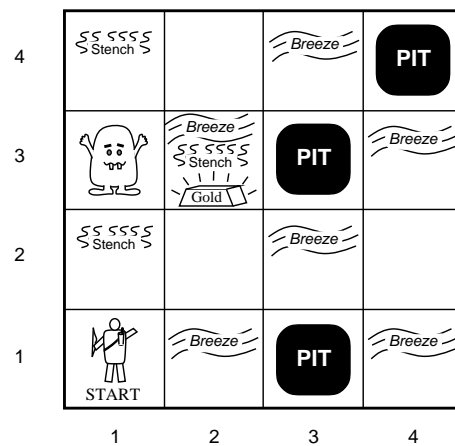
- What is the best sequence of actions to carry out to get the gold?

Summary

- This lecture started with probabilistic inference.

- However, the main work of the night was to do with utilities.

- We now know how to make a decision about the best action to carry out.

  - But we canonly choose one action at a time.

- Next time we'll look at sequential decision problems.