GAME THEORY II

- A two-person general sum game is represented by two matrices A and B.

- For instance:
$$A = \begin{bmatrix} -3 & 0 \\ -4 & -1 \end{bmatrix}$$
$$B = \begin{bmatrix} -3 & -4 \\ 0 & -1 \end{bmatrix}$$

- a_{ij} is the payoff to P1 and b_{ij} is the payoff to P2.

- If:
$$a_{ij} = -b_{ij}$$

then we have a zero-sum game.

- We often write the payoff matrix as:
$$(A, B) = \begin{bmatrix} (-3, -3) & (0, -4) \\ (-4, 0) & (-1, -1) \end{bmatrix}$$

where each entry in the matrix is (a_{ij}, b_{ij})

- A strategy (i^*, j^*) is a *Nash equilibrium solution* to the game (A, B) if:
$$\forall i, a_{i^*, j^*} \geq a_{i, j^*}$$
$$\forall j, b_{i^*, j^*} \geq b_{i^*, j}$$

- If both players play a Nash equilibrium strategy, then neither can unilaterally move away from the strategy and profit.

- The Nash equilibrium is a generalisation of the idea of a saddle point in a zero-sum game.

- Note that not every game has a Nash equilibrium for pure strategies.

- Note also that A Nash solution:
  - need not be the best solution; and
  - need not be a resonable solution.

- All the Nash equilibrium guarantees is stability.

- This stability amounts to protection against exploitation by the other player.

- There may be more than one Nash equilibrium.

- In the game:
$$(A, B) = \begin{bmatrix} (4,0) & (4,1) \\ (5,3) & (3,2) \end{bmatrix}$$
  there are two Nash equilibrium strategies:
$$(4, 1)$$
  and
$$(5, 3)$$
- Clearly both players would prefer the second to the first.

- A good thing about Nash equilibrium, is that every two-player general-sum game has a Nash equilibrium solution.
- However, these need not be pure strategy solutions.
- To ensure we can find a Nash equilibrium, we have to look for mixed strategies.
- For mixed strategies, as for zero sum games, each player is looking for a probability vector.
- P1 is looking for:
$$x = (x_1, x_2)$$
- P2 is looking for:
$$y = (y_1, y_2)$$

- For a game with payoff matrices A and B, a mixed strategy (x^*, y^*) is a Nash equilibrium solution if:

$$\forall x, x^* A y^{*T} \geq x A y^{*T}$$
$$\forall y, x^* B y^{*T} \geq x B y^{*T}$$

- In other words, x^* gives a higher expected value to P1 than any other strategy when P2 plays y^*.
- Similarly, y^* gives a higher expected value to P2 than any other strategy when P1 plays x^*.

The Prisoner's Dilemma

- Two suspects have been arrested by the police and are being questioned separately.
- If they both say nothing, they will be sentenced to 1 year $(-1, -1)$ on existing evidence.
- If either incriminates the other while the other remains silent, the incriminator will be released, and their co-suspect will be sent to prison for 4 years $(0, -4)$, $(-4, 0)$.
- If both incriminate each other, they will both be sentenced to 3 years $(-3, -3)$.

- This gives the game:
$$(A, B) = \begin{bmatrix} (-3, -3) & (0, -4) \\ (-4, 0) & (-1, -1) \end{bmatrix}$$

- Here the Nash equilibrium strategy is $(-3, -3)$.

- It is stable because it is not risky for either player—if you confess then there is nothing your co-suspect can do to make you worse off.

- However, this is a much worse outcome than if both players refused to say anything.

- Such a strategy, though, would be very risky and so unstable.

- In this case, the Nash equilibrium strategy contrasts with the Pareto efficient outcome.

- A solution is *Pareto efficient* if there is no other outcome which makes one play better off and doesn't make the second player worse off.

- Here the solution $(-1, -1)$ is Pareto efficient.

- One way to think about this is that Nash equilibrium strategies give the best outcome for an individual.

- In contrast, Pareto efficient solutions give the best outcome for all players together.

- These results seems to suggest that in looking for Nash equilibrium solutions:
  - It is natural to defect; and
  - Defection gives sub-optimal behvaiour.
- Both of these are true to some extent.
- "Sub-optimal" is true if we are looking to maximise utility.
- If we are looking to minimise risk, then the the Nash equilibrium solution is optimal.

- There are situations in which it is natural to co-operate.
- If the game is played several times, and players have memory of what happened on the previous round then:
  - Defection can be punished; and
  - The sucker's payoff can be amortised.
- Provided that the "shadow of the future" is big enough, these factors encourage co-operative behaviour.
- In the "Iterated Prisoner's Dilemma" strategies like "Tit for tat" do well.

- There are other some other scenarios, similar to the Prisoner's Dilemma, which are interesting to consider:
- These are:
  - Co-operation dominates
  - Defection dominates
  - The stag hunt
  - The game of chicken.
- All of these use a payoff matrix that is very similar to that above.

- If the payoff matrix is:

$$(A, B) = \begin{bmatrix} (-3, -3) & (-1, -1) \\ (-1, -1) & (0, 0) \end{bmatrix}$$

Then co-operation dominates

- If the payoff matrix is:

$$(A, B) = \begin{bmatrix} (0, 0) & (-1, -1) \\ (-1, -1) & (-3, -3) \end{bmatrix}$$

Then defection dominates.

The stag hunt scenario can be described by:

You and a friend decide it would be a great joke to show up on the last day of school with some ridiculous haircut. Egged on by your clique, you both *swear* you'll get the haircut.

A night of indecision follows. As you anticipate your parents' and teachers reactions [. . .] you start wondering if your friend is really going to go through with the plan.

Not that you don't want the plan to succeed: the best possible outcome would be for both of you to get the haircut.

The trouble is, it would be awful to be the *only* one to show up with the haircut. That would be the worst possible outcome.

You're not above enjoying your friend's embarrassment. If you *didn't* get the haircut, but the friend did, and looked like a real jerk, that would be almost as good as if you both got the haircut.

- A payoff matrix like:

$$(A, B) = \begin{bmatrix} (-3, -3) & (-1, -4) \\ (-4, -1) & (0, 0) \end{bmatrix}$$

  describes this scenario.

- The difference from the prisoner's dilemma is that now it is better if you both co-operate than if you defect while the other co-operates.

- There are two Nash equilibrium solutions:

  – Both co-operate
  – Both defect

The game of chicken gets its name from a rather silly, macho "game" that was supposedly popular amongst juvenile delinquents in 1950s America; the game was immortalised by James Dean in the 1950s film *Rebel without a Cause*. The purpose of the game is to establish who is bravest of the two players.

The game is played by both players driving their cars at high speed towards a cliff. The idea is that the least brave of the two (the "chicken") will be the first to drop out of the game by jumping out of the speeding car. The winner is the one who lasts longest in the car. Of course, if *neither* player jumps out of the car, then both cars fly off the cliff, taking their foolish passengers to a fiery death on the rocks that undoubtedly lie at the foot of the cliff.

- Chicken has a payoff matrix like:

$$(A, B) = \begin{bmatrix} (-4, -4) & (0, -3) \\ (-3, 0) & (-1, -1) \end{bmatrix}$$

  co-operation is taken to be jumping out of the car.

- This differs from the Prisoner's Dilemma in that both defecting is the wrost possible outcome.

- There are two Nash equilibrium solutions:

  – P1 co-operates and P2 defects.
  – P2 co-operates and P1 defects.

Summary

- This lecture has introduced the idea of Nash equilibrium

- It has considered how Nash equilibrium may be used to analyse a number of canonical games.

- The notion of Pareto optimality was also discussed.