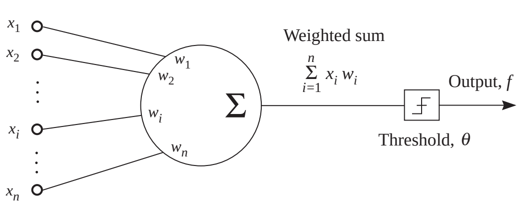
# NEURAL NETWORKS II

## Introduction

- This lecture builds on the description in the previous lecture to establish how to train neural networks.
- We will work out a general approach.
- We will then give three particular versions that are commonly used.
- We start with a quick recap.



## Recap






- Output is 1 if $s = X \cdot W > \theta$
- Output is 0 otherwise



## Gradient descent methods

- A common way to train a TLU is through an error function.
- We define:

$$\epsilon = \sum_{X \in \Theta} (d_i(X_i) - f_i(X_i))^2$$

- where:
  - $d_i(X_i)$ is the outcome we want for $X_i$;
  - $f_i(X_i)$ is the outcome we get.
- Often we write these functions as $d_i$ and $f_i$.
- We then minimise $\epsilon$

- If $\theta$ is rolled into the weights, then the value of $\epsilon$ depends on the weights.
- (Since these determine the value of $f_i$.)
- We minimise by looking at the gradient of $\epsilon$ with respect to the weights...
- ...and then altering the weights to move $\epsilon$ down the gradient.
- Eventually this *gradient descent* will take us down to the minimum value of $\epsilon$.



- The computation of $\epsilon$ is complicated by the fact that its value depends on *all* the $X_i$ in $\Theta$.
- Often it is easier to do the calculation for one $X_i$, adjust the weights to move down the gradient, and then start over with another $X_j$.
- Thus we do the learning incrementally, taking each member of $\Theta$ in an order $\Sigma$.
- The incremental version only ever approximates the result of doing it "properly" (the batch way), but often it is a good approximation.
- Here we will just look at the incremental version.



- When we have a single input vector $X$, with output $f$ and desired output $d$, the error is:

$$\epsilon = (d - f)^2$$

- The gradient of $\epsilon$ with respect to the weights is

$$\frac{\partial \epsilon}{\partial W}$$

and

$$\frac{\partial \epsilon}{\partial W} = \left[ \frac{\partial \epsilon}{\partial w_1}, \frac{\partial \epsilon}{\partial w_2}, \ldots, \frac{\partial \epsilon}{\partial w_{n+1}} \right]$$



- Since $\epsilon$ depends on $W$ through

$$s = X \cdot W$$

it follows that:

$$\frac{\partial \epsilon}{\partial W} = \frac{\partial \epsilon}{\partial s} \cdot \frac{\partial s}{\partial W}$$

- Since:

$$\frac{\partial s}{\partial W} = X$$

it follows that:

$$\frac{\partial \epsilon}{\partial W} = \frac{\partial \epsilon}{\partial s} X$$

- Furthermore we can write:
$$\frac{\partial \epsilon}{\partial s} = -2(d-f)\frac{\partial f}{\partial s}$$
and so:
$$\frac{\partial \epsilon}{\partial W} = -2(d-f)\frac{\partial f}{\partial s}X$$
- This seems to give us a way of working out what the gradient is.
- However, we have a problem.



- The problem is that the TLU output $f$, cannot be differentiated.
- Most times we vary $s$ a little we get no change in $f$.
- Sometimes, though, we get a big change (from 0 to 1 or vice-versa).
- There are several ways around this.
  - Ignore the threshold and set $f = s$.
  - Replace the threshold function with something we can differentiate or otherwise find the gradient of.
- We will look at both of these.



## The Widrow-Hoff procedure

- Let's try and adjust the weights so that:
  - Every training vector labelled with a 1 produces a dot product of 1; and
  - Every training vector labelled with a 0 produces a dot product of -1.
- Then, with
$$f = s$$
the incremental squared error is:
$$\epsilon = (d-f)^2 = (d-s)^2$$
and
$$\frac{\partial f}{\partial s} = 1$$



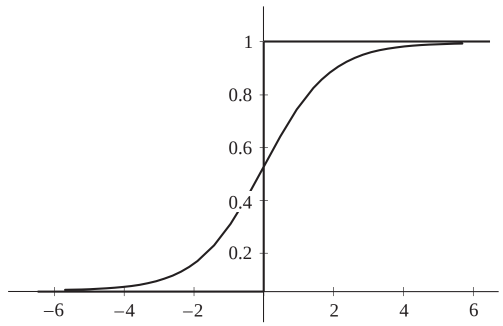
- This makes the gradient:
$$\frac{\partial \epsilon}{\partial W} = -2(d-f)X$$
- If we want to then move the weight vector down the gradient, we can set the new value of the weight vector as:
$$W := W + c(d-f)X$$
- The factor of 2 is combined into the *learning rate parameter* $c$.
- As always this controls the speed of the adjustment by determining the fraction of $X$ added to $W$.

- Whenever the error:
$$(d - f)$$
is positive, then we add a fraction of the input into the weight.
- This increases $X \cdot W$, and so decreases
$$(d - f)$$
- If the error is negative we subtract a fraction of the input and reverse the effect.
- Once we have found the best set of weights, we can go back to using the threshold function.



## The generalised Delta procedure

- Another way to handle the threshold function is to replace it with something we can differentiate.








- This function is known as a *sigmoid*:
$$f(s) = \frac{1}{1 + e^{-s}}$$
- With this function, we have the partial derivative:
$$\frac{\partial f}{\partial s} = f(1 - f)$$
- Since
$$\frac{\partial \epsilon}{\partial W} = -2(d - f)\frac{\partial f}{\partial s}X$$
we have:
$$\frac{\partial \epsilon}{\partial W} = -2(d - f)f(1 - f)X$$



- This gives us another rule for changing weights:
$$W := W + c(d - f)f(1 - f)X$$
- This compares to the Widrow-Hoff procedure as follows:
  - In W-H, $d$ is 1 or -1. In generalised Delta it is 1 or 0.
  - In W-H, $f$ is equal to $s$. In generalised Delta, $f$ is the output of the sigmoid function.
  - Generalised Delta has the extra term $f(1 - f)$
- With the sigmoid, $f(1 - f)$ varies in value from 0 to 1.
- It has value 0 when $f$ is 0 or 1.
- It has maximum value of 0.25 when $f$ has value 0.5 (and the input to the sigmoid is 0).

- One can think of the sigmoid as a "fuzzy boundary".
- When the input is a long way from the boundary, $f(1-f)$ has a value close to 0.
- Thus hardly any adjustment is made to the weights.
- When the input is closer to the boundary, then weight changes are more significant.
- These changes are always to reduce the error.
- Once the weights are established, we can go back to using the step function.



## A general approach

- Both these techniques have done the same thing.
- They have replaced something we couldn't find the slope of with something we could.
- We could do the same with a different gradient function.
- This obviously trains the weights approximately.
- However, it seems that the approximation is often good enough.
- In any case, we are interested in performance on non-training examples.



## The error-correction procedure

- Another approach keeps the original threshold function.
- We then forget about differentiation and just adjust the weights when the TLU gives a classification error.
- In other words we make a change when:

$$(d - f)$$

  has value 1 or -1.
- This time the weight change rule is:

$$W := W + c(d - f)X$$

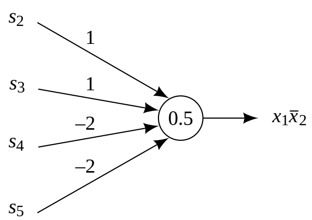
- Just as before, the change tends to reduce the error.



- Comparing this with Widrow-Hoff, we note that both $d$ and $f$ are either 0 or 1.
- Whereas in W-H, $d$ is 1 or $-1$ and $f = s$.
- It is possible to prove that if there is a $W$ that gives a correct output for all $X \in \Theta$,
- Then after a finite number of adjustments, this error-correction procedure will find this weight vector.
- Thus the process will terminate, making no more weight adjustments.
- For nonlinearly separable sets of input vectors, the procedure will not terminate (as opposed to W-H and generalised Delta).

- Since (as we saw last lecture) the rules/network for the boundary following robot are linearly separable functions…
- …we can use any of these procedures to learn the weights for a TLU to implement these functions, such as:

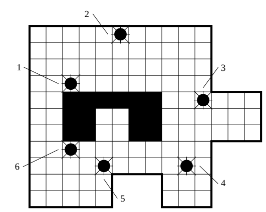
$$(s_2 + s_3)\overline{s_4}.\overline{s_5}$$








- A suitable training set for training this TLU is




| Input number | Sensory vector | $x_1\overline{x_2}$ (move east) |
|---|---|---|
| 1 | 00001100 | 0 |
| 2 | 11100000 | 1 |
| 3 | 00100000 | 1 |
| 4 | 00000000 | 0 |
| 5 | 00001000 | 0 |
| 6 | 01100000 | 1 |





## Summary

- In this lecture we looked at methods for training TLUs.
- All the methods were *gradient descent*—they adjusted weights to reduce the error, step-by-step.
- They differed in what they used for the threshold function.
- Widrow-Hoff ignores it and sets $f = s$.
- Generalised-delta uses a function that can be differentiated.
- Error-correction uses the step function.