Arrow's Impossibility Theorem

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Arrow's Theorem and the Gibbard - Satterthwaite Theorem: A Unified Approach, June 2000 version.

Notation

A : Finite set of alternatives.
L : Set of strict linear orders on A.
L^* : Set of weak linear orders on A.

Fix a positive integer N.

A function $f : L^N \to A$ will be called a **social choice function**.
A function $F : L^N \to L^*$ will be called a **social welfare function**.

A member of L^N is called a **profile**, and its ith component is called **individual i's ranking**.
A member of L^* is called a **social order**, or **society's ranking**.

Pareto Efficiency

We say that a social choice function $f : L^N \to A$ is **pareto efficient** if whenever alternative a is at the top of every individual i's ranking, L_i, then $f(L_1, \ldots, L_N) = a$.

L_1	L_2	$\ldots$	L_i	$\ldots$	L_{N-1}	L_N	**Social Choice**
a	a	$\ldots$	a	$\ldots$	a	a	a
.	.	$\ldots$	.	$\ldots$	.	.	
.	.	$\ldots$	.	$\ldots$	.	.	
.	.	$\ldots$	.	$\ldots$	.	.	
.	.	$\ldots$	.	$\ldots$	.	.	
.	.	$\ldots$	.	$\ldots$	.	.	

Pareto Efficiency

We say that a social welfare function $f : L^N \to L^*$ is **pareto efficient** if whenever alternative a is ranked above b according to each L_i, then a is ranked above b according to $F(L_1,\ldots,L_N)$.

L_1	L_2	$\ldots$	L_i	$\ldots$	L_{N-1}	L_N	**Social Order**
.	a	$\ldots$	.	$\ldots$	.	.	.
.	.	$\ldots$	.	$\ldots$	.	a	.
a	b	$\ldots$	.	$\ldots$	.	.	.
b	.	$\ldots$	a	$\ldots$	a	.	a
.	.	$\ldots$	.	$\ldots$	b	.	.
.	.	$\ldots$	b	$\ldots$	.	b	b

Monotonicity

We say that a social choice function $f : L^N \rightarrow A$ is **monotonic** if whenever $f(L_1, \ldots, L_N) = a$ and for every individual i and every alternative b the ranking $L_i^{'}$ ranks a above b if L_i does, then $f(L_1^{'}, \ldots, L_N^{'}) = a$.

Independency of Irrelevant Alternatives

We say that a social welfare function $f : L^N \to L^*$ is **independent of irrelevant alternatives** if whenever the ranking of a versus b is unchanged for each $i = 1, \ldots, N$ when individual i's ranking changes from L_i to L_i', then the ranking of a versus b is the same according to both $F(L_1, \ldots, L_N)$ and $F(L_1', \ldots, L_N')$.

Dictatorship

We say that a social choice function $f : L^N \to A$ is **dictatorial** if there is an individual i such that $f(L_1, \ldots, L_N) = a$ if and only if a is at the top of i's ranking L_i.

Dictatorship

We say that a social welfare function $f : L^N \to L^*$ is **dictatorial** if there is an individual i such that one alternative is ranked above another according to $F(L_1,...,L_N)$ whenever the one is ranked above the other according to L_i.

Theorems

Theorem A (a version of Muller – Satterthwaite theorem): If $\# A \geq 3$ and $f : L^N \to A$ is pareto-efficient and monotonic, then f is a dictatorial social choice function.

Theorem B (Arrow's theorem): If $\# A \geq 3$ and $F : L^N \to L^*$ satisfies pareto-efficiency and IIA, then F is a dictatorial social welfare function.

Proof

Step 1. Consider any two distinct alternatives $a, b \in A$ and a profile of rankings in which a is ranked the highest and b lowest for every individual $i = 1, \ldots, N$.

Social Choice: Pareto efficiency implies that the social choice at this profile is a.
Social Order: Pareto efficiency implies that a is strictly at the top of the social order.

L_1	…	L_i	…	L_{N-1}	L_N	**Social Choice**	**Social Order**
a	…	a	…	a	a	a	a
.	…	.	…	.	.		.
.	…	.	…	.	.		.
.	…	.	…	.	.		.
.	…	.	…	.	.		.
b	…	b	…	b	b		.

Proof

Now change individual 1's ranking by raising b in it by one position at a time.

Social Choice: By monotonicity, the social choice remains equal to a so long as b is below a in 1's ranking.
But when b finally does rise above a, monotonicity implies that the social choice

- either changes to b
- or remains equal to a.

Social Order: By IIA, a remains at the top of the social order so long as b is below a in 1's ranking.
But when b finally does rise above a, IIA implies that a remains ranked above every alternative but perhaps b by the social order.

Proof

Social Choice: If the social choice remains equal to a, then begin the same process with individual 2, then 3, etc. until for some individual j, the social choice does change from a to b when b rises above a in j's ranking.

There must be such an individual j because alternative b will eventually be at the top of every individual's ranking and by pareto efficiency the social choice will then be b.

Social Order: If a does remain ranked above b, then begin the same process with individual 2, then 3, etc. until for some individual j, the social rank of b rises above a when b rises above a in j's ranking.

There must be such an individual j because alternative b will eventually be at the top of every individual's ranking and by pareto efficiency b then will be ranked above a.

Before:

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
b	…	b	a	a	…	a	a	a
a	…	a	b	.	…	.		.
.	…	.	.	.	…	.		.
.	…	.	.	.	…	.		b
.	…	.	.	.	…	.		.
.	…	.	.	b	…	b		.

Figure 1

After:

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
b	…	b	b	a	…	a	b	b
a	…	a	a	.	…	.		a
.	…	.	.	.	…	.		.
.	…	.	.	.	…	.		.
.	…	.	.	.	…	.		.
.	…	.	.	b	…	b		.

Figure 2

Proof

Step 2. Next we will obtain Figures 1' and 2' by moving the alternative a to the bottom of individual i's ranking for $i < j$ and moving it to the second last position i's ranking for $i > j$ in Figures 1 and 2.

Social Choice: These changes should not affect the social choices in Figures 1 and 2.

Social Order: These changes should not affect the top-ranked alternatives and the social orders are as follows:

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
b	…	b	a	.	…	.	a	a
.	…	.	b	.	…	.		b
.	…	.	.	.	…	.		.
.	…	.	.	.	…	.		.
.	…	.	.	a	…	a		.
a	…	a	.	b	…	b		.

Figure 1’

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
b	…	b	b	.	…	.	b	b
.	…	.	a	.	…	.		a
.	…	.	.	.	…	.		.
.	…	.	.	.	…	.		.
.	…	.	.	a	…	a		.
a	…	a	.	b	…	b		.

Figure 2’

Proof

Social Choice: The social choice in Figure 2', by monotonicity, must be b because the social choice in Figure 2 is b and no individual's ranking of b versus any alternative changes in the move from Figure 2 to Figure 2'.

The profiles in Figure 1' and 2' differ only in individual j's ranking of alternatives a and b. So, because the social choice in Figure 2' is b, the social choice in Figure 1' must, by monotonicity, be either a or b.

But if the social choice in Figure 1' is b, then by monotonicity, the social choice in Figure 1 must also be b, a contradiction. That means the social choice in Figure 1' is a.

Proof

Social Order: In Figure 2', by IIA, the top-ranked alternative must be b because it is top-ranked in Figure 2 and no individual's ranking of b versus any alternative changes in the move from Figure 2 to Figure 2'.

The profiles in Figure 1' and 2' differ only in individual j's ranking of alternatives a and b. So, by IIA, in Figure 1', b must remain ranked above every alternative but perhaps a.

But if b is ranked at least as high as a in Figure 1', then by IIA, b would also be socially ranked as high as a in Figure 1, a contradiction. That means a is ranked first and b second in Figure 1'.

Proof

Step 3. Consider $c \in A$ distinct from a and b.

Because the profile of rankings in Figure 3 can be obtained from the Figure 1’ without changing the ranking of a versus any other alternative in any individual’s ranking:

- **Social Choice:** the social choice in Figure 3 must, by monotonicity be a.
- **Social Order:** society’s top-ranked choice in Figure 3 must, by IIA, be a.

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
b	…	b	a	.	…	.	a	a
.	…	.	b	.	…	.		b
.	…	.	.	.	…	.		.
.	…	.	.	.	…	.		.
.	…	.	.	a	…	a		.
a	…	a	.	b	…	b		.

Figure 1'

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
.	…	.	a	.	…	.	a	a
.	…	.	c	.	…	.		.
.	…	.	b	.	…	.		.
c	…	c	.	c	…	c		.
b	…	b	.	a	…	a		.
a	…	a	.	b	…	b		.

Figure 3

Proof

Step 4. Next we will obtain the profile in Figure 4 by interchanging the ranking of alternatives a and b for individuals $i > j$.

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
.	…	.	a	.	…	.	a	a
.	…	.	c	.	…	.		.
.	…	.	b	.	…	.		.
c	…	c	.	c	…	c		.
b	…	b	.	a	…	a		.
a	…	a	.	b	…	b		.

Figure 3

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
.	…	.	a	.	…	.	?	?
.	…	.	c	.	…	.		.
.	…	.	b	.	…	.		.
c	…	c	.	c	…	c		.
b	…	b	.	b	…	b		.
a	…	a	.	a	…	a		.

Figure 4

Proof

Social Choice: Since the only difference between Figure 3 and 4 is the ranking of alternatives a and b for $i > j$, and because the social choice in Figure 3 is a, the social choice in Figure 4 must, by monotonicity, be either a or b.

But the social choice in Figure 4 cannot be b because alternative c is ranked above b in every individual's Figure 4 ranking, and monotonicity would then imply that the social choice would remain b even if c were raised to the top of every individual's ranking, contradicting pareto efficiency. Then the social choice in Figure 4 must be a.

Social Order: Since the only difference between Figure 3 and 4 is the ranking of alternatives a and b for $i > j$, and because a is top-ranked in Figure 3, IIA implies that the ranking of a remains above c as well as every other alternative, but perhaps b in Figure 4.

But because every alternative is ranked above b in every individual's Figure 4 ranking, the social ranking of c must be above b by pareto efficiency. Then a is top-ranked and c is ranked above b in Figure 4.

Proof

L_1	…	L_{j-1}	L_j	L_{j+1}	…	L_N	**Social Choice**	**Social Order**
.	…	.	a	.	…	.	a	a
.	…	.	c	.	…	.		.
.	…	.	b	.	…	.		c
c	…	c	.	c	…	c		.
b	…	b	.	b	…	b		b
a	…	a	.	a	…	a		.

Figure 4

Proof: Social Choice

Step 5. An arbitrary profile of rankings with a at the top of individual j's ranking can be obtained from the profile in Figure 4 without reducing the ranking of a versus any other alternative in any individual's ranking.

Monotonicity implies that the social choice must be a whenever a is at the top of individual j's ranking. So, we may say that individual j is a dictator for alternative a:

Because a was arbitrary, for each alternative $a \in A$, there is a dictator for a. But there cannot be distinct dictators for distinct alternatives. Hence there is a single dictator for all alternatives.

Proof: Social Order

Step 5. Consider an arbitrary profile of rankings with a above b in individual j's ranking. If necessary, alter the profile by moving alternative c between a and b in j's ranking and to the top of every other individual's ranking. By IIA this does not affect the ranking of a versus b. Because the ranking of a versus c for every individual is now as in Figure 4, IIA implies that the ranking of a is above c, which by Pareto efficiency is ranked above b. So, by transitivity, we may conclude that a is ranked above b whenever j ranks a above b.

By repeating the argument with the roles of b and c reversed, and recalling that c was an arbitrary alternative distinct from a and b, we may conclude that the social ranking of a is above some alternative whenever j ranks a above that alternative. So, we may say that individual j is a dictator for a.

Since a was an arbitrary alternative we have shown that for every alternative $a \in A$, there is a dictator for a. But clearly there cannot be distinct dictators for distinct alternatives. Hence there is a single dictator for all alternatives.