Fair Division

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Outlook of the Talk

- Introduction to Social Choice Theory
- Fair Division Algorithms
- Some Remarks

Earlier Works

Social choice theory is "concerned with relationships between individuals' preferences and social choice".

Arrow notes that "in a *capitalist democracy*, there are essentially two methods by which social choices are made: voting, typically used to make "political" decisions, and the market mechanism, typically used to make "economic" decisions."[1].

Sen writes

> *While Aristotle agreed with Agathon that even God could not change the past, he did think that the future was ours to make - by basing our choices on reasoning.*

[1]emphasize is mine - C.B.

Assumptions

- Utility should be measurable and interpersonally comparable.
- Agents should be rational.
- Objects of choice are social states.

(Arrow, *Social Choice and Individual Values*)
(Sen, Choice, Welfare and Measurement)
Notice that all assumptions (not, *axioms*) are open to debate.

Observations

- Preference precedes the choice.
- Choice is consistent, hence *transitive*
- Arrow defined a “social welfare function” as a relation that specifies a social ordering over all social states for every set of individual preference orderings.
- Thus, as Sen remarked, “a camel is a horse designed by a committee”.
- Social Choice Theory existed in non-capitalist world, too. Possibility of cyclical paths, USSR

What Can We Compute?

- Preference aggregations for fairness
- Voting methods for better democracy
- Games and their complexities for efficiency
- Computational complexity of social procedures
- Resource allocation for social welfare
- Coalitions for more democracy
- Social software for facebook haters

Cut and Choose

How can a homogenous cake among two players be fairly divided?

One cuts, the other chooses.

"Notice that each child's strategy guarantees him or her 'satisfaction', regardless of what the other child does."

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Envy-freeness

A precedure is called “envy-free” if each player receives a piece s/he would not swap for that received by any other player. In this case, no player is envious of the piece of any other player.
We can extend the definition to a game theoretical setting. A protocol is called “envy-free” if each of the n participants has a strategy that will guarantee him a piece that is, according to his own measure, at least tied for largest.

Steinhaus Protocol for Three Players I

1. Player 1 cuts the cake into 3 pieces (that he considers to be the same size).
2. Player 2 is given the choice of either passing, i.e., doing nothing (which he does if he thinks 2 or more of the pieces are of size at least $1/3$), or not passing and labeling 2 of the pieces (that he thinks are of size strictly less than $1/3$) as "bad."

Steinhaus Protocol for Three Players II

3. If Player 2 passed in step 2, then Players 3, 2, and 1, in that order, choose a piece (that they consider to be of size at least 1/3).
   Aside: In this case, each player receives a piece of size at least 1/3 in his own measure. This is true of: Player 3, because he chooses first; Player 2, because he thinks either 2 or 3 pieces are that large, and so at least one of them will still be available after Player 3 chooses his piece; and Player 1, because he made all 3 pieces of size 1/3.
4. If Player 2 did not pass at Step 2, then Player 3 is given the same two options that Player 2 had at Step 2. He ignores Player 2's labels.

Steinhaus Protocol for Three Players III

5. If Player 3 passed in Step 4, then Players 2, 3, and 1, in that order, choose a piece (that they consider to be of size at least 1/3).
   Aside: In this case, as before, each player receives a piece of size at least 1/3 of his own measure.
6. If Player 3 did not pass at Step 4, then Player 1 is required to take a piece that both Player 2 and Player 3 labelled as "bad."
   Aside: Note first that there certainly must be such a piece. At this point, Player 1 has received a piece that he thinks is of size exactly 1/3, which both Player 1 and Player 2 think is "bad," i.e., of size strictly less than 1/3.

Steinhaus Protocol for Three Players IV

7. The other two pieces are reassembled, and Player 2 cuts the resulting piece into two pieces (that he considers to be the same size).
8. Player 3 chooses one of the two pieces (that he considers to be at least tied for largest).
9. Player 2 is given the remaining piece.
   Aside: This is just cut-and-choose between Players 2 and 3, which ends the protocol.

Dubins-Spanier Scheme

A knife is slowly moved along the top of the cake so that all the slices made are parallel. Each player calls "cut" when he or she is willing to take the resulting piece as his or her allocation. Similar to Dutch auctions. Notice that this scheme is not an *algorithm*.

Banach - Knasser Scheme for n players I

1. Player 1 cuts a piece P_1 (of size $1/n$) from the cake.
2. Player 2 is given the choice of either passing (which he does if he thinks P_1 is of size less than $1/n$), or trimming a piece from P_1 to create a smaller piece (that he thinks is of size exactly $1/n$). The piece P_1, now perhaps trimmed, is renamed P_2. The trimmings are set aside.
3. For $3 \leq i \leq n$, Player i takes the piece P_{i-1} and proceeds exactly as Player 2 did in Step 2, with the resulting piece now called P_i.

   Aside: For $1 \leq i \leq n$, Player i thinks that P_i is of size less than or equal to $1/n$. We also have that $P_1 \supset \cdots \supset P_n$. Thus, every player thinks Pn is of size at most $1/n$.

Banach - Knasser Scheme for n players II

4. The last player to trim the piece, or Player 1 if no one trimmed it, is given P_n.
   Aside: The player receiving P_n thinks it is of size exactly $1/n$.
5. The trimmings are reassembled, and Steps 1-4 are repeated for the remainder of the cake, and with the remaining $n-1$ players in place of the original n players.
   Aside: The player who gets a piece at this second stage is getting exactly $1/(n-1)$ of the remainder of the cake; he, and everyone else, thinks this remainder is of size at least $(n-1)/n$. Hence, he thinks his piece is of size at least $1/n$.

Banach - Knasser Scheme for n players III

6. Step 5 is iterated until there are only 2 players left. The last 2 players use cut-and-choose.
   Aside: As before, each player receives a piece that he thinks is of size at least $1/n$. This ends the protocol.

Remarks

Brams - Taylor algorithm is complicated compared to the classical ones.

Fair division algorithms are used in resource allocation problems in networks, grid computing, parallel programming.

Variety of political science applications are also suggested.

For Further Reading

- Amartya Sen, The Impossibility of a Paretian Liberal, 1970.
- Amartya Sen, Choice, Welfare and Measurement, 1997.
- Amartya Sen, The Possibility of Social Choice, 1999.
- Aleskerov, The History of Social Choice in Russia and Soviet Union
- Ulle Endriss et al., A Short Introduction to Computational Social Choice.
- Kenneth Arrow, Social Choice and Individual Values
- Steven Brams and Alan Taylor, An Envy-Free Cake Division Protocol, 1995.
- Dubins and Spanier, How to cut a cake fairly, 1961.

Thanks for your attention!

For the slides:

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