Social Choice Theory
- Voting Systems

Condorcet's Paradox

The foundations of social choice theory lie in the "Paradox of Voting."

Let X = {a, b, c} be a set of alternatives. Consider a society that consists of three members called 1, 2, and 3. Their rankings of *X* are:

$a >_1 b >_1 c,$

$b >_2 c >_2 a,$

$c >_3 a >_3 b.$

The **Condorcet's paradox** (also known as **paradox of voting**) is a situation in which collective preferences can be cyclic (i.e. not transitive), even if the preferences of individual voters are not. This is paradoxical, because it means that majority wishes can be in conflict with each other.

Social Choice Theory

Social choice theory is related to the design of voting systems, which are methods for determining social action on the basis of individuals' preferences.

A basic model of social choice consists of the following:

- X: a set of social alternatives.
- N: a finite set of individuals (denote the number of elements in N by n).
- i: individual i's linear ordering on X.
- Profile: An n-tuple of orderings (1, . . . ,n) interpreted as a certain "state of society."
- Social Choice Function, attaches a social alternative, interpreted as the society's choice, to every profile of preference relations.
- SWF (Social Welfare Function): A function that assigns a single (social) preference (not necessarily a linear ordering) to every profile.

Some Examples of Aggregation Procedures

1. Plurality: simple plurality, first past the post or winner-takes-all.

   In this voting system the single winner is the person with the most votes; there is no requirement that the winner gain an absolute majority of votes.

2. Dictatorship: There is an individual i (the dictator) so that x is preferred to y if and only if $x >_i y$.

3. Anti-dictatorship: There is an individual i so that x is preferred to y if and only if $y >_i x$.

4. Borda count: The voters rank candidates in order of preference. The Borda count determines the winner of an election by giving each candidate a certain number of points corresponding to the position in which he/she is ranked by each voter. Once all votes have been counted the candidate with the most points is the winner.

Condition Pareto (Par):

The Pareto axiom requires that if all individuals prefer one alternative over the other, then the social preferences agree with the individuals'.

Condition IIA (Independence of Irrelevant Alternatives):

In voting systems, independence of irrelevant alternatives is often interpreted as, if one candidate (X) wins the election, and a new alternative (Y) is added, only X or Y will win the election.

A society's choice between any A and B should be a function of only the choices of the individuals in the group between A and B. In particular, it should not change if some individuals in the group change their minds about C and/or D.

Plurality contradicts IIA

Example:

In a plurality voting system 7 voters rank 3 alternatives [A, B, C].

3 voters rank [A>B>C]

2 voters rank [B>A>C]

2 voters rank [C>B>A]

The plurality winner is A, with 3 votes out of 7.

Now, the 2 voters who rank [C>B>A] instead rank [B>C>A].

They only change their preferences over [B, C].

The new plurality winner is B, with 4 votes out of 7.

The social choice has swapped B ahead of A, even though the only change in the preference profile was between B and C, which are irrelevant alternatives.

Borda count contradicts IIA

Example:

In a Borda count voting system, 5 voters rank 5 alternatives [A, B, C, D, E].

3 voters rank [A>B>C>D>E].

1 voter ranks [C>D>E>B>A].

1 voter ranks [E>C>D>B>A].

Points: (4, 3, 2, 1, 0).

Borda count: C=13, A=12, B=11, D=8, E=6. C wins.

Now, the voter who ranks [C>D>E>B>A] instead ranks [C>B>E>D>A]; and the voter who ranks [E>C>D>B>A] instead ranks [E>C>B>D>A].

They change their preferences only over the pairs [B, D] and [B, E].

The new Borda count: B=14, C=13, A=12, E=6, D=5. B wins.

The social choice has changed the ranking of [B, A], [B, C] and [D, E]. The changes in the social choice ranking are dependent on irrelevant changes in the preference profile. In particular, B now wins instead of C, even though no voter changed their preference over [B, C].

Anti-dictatorship contradicts par

Example:

In an anti-dictatorship voting system, 10 voters rank 5 alternatives [A, B, C, D, E].

5 voters rank [A>B>E>D>C].
4 voters rank [A>D>C>E>B].
1 voter (the anti-dictator) ranks [A>B>C>D>E].

E wins even though every individual preferred A over E.