Connecting Lexicographic with Maximum Entropy Entailment

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Abstract. This paper reviews and relates two default reasoning mechanisms, lexicographic (lex) and maximum entropy (me) entailment. Me-entailment requires that defaults be assigned specific strengths and it is shown that lex-entailment can be equated to me-entailment for a class of specific strength assignments. By clarifying the assumptions which underlie lex-entailment, it is argued that me-entailment is a superior method of handling default inference for reasons of both expressiveness and objective justification.

1 Introduction

The most widely accepted extension to a set of defaults is its p-closure [6] which is the fixed point result of applying the rules of System P. The p-closure contains all defaults which can be probabilistically entailed in the sense of Adams [1]. But the p-closure is too conservative to sanction common patterns of nonmonotonic reasoning such as the ability to ignore irrelevant information or to allow inheritance to exceptional subclasses. Lehmann and Magidor's rational closure [8], or equivalently Pearl's System Z [10], succeeded in solving the first problem but the inheritance problem requires more sophisticated machinery.

This paper examines two systems which have been proposed to deal with the exceptional inheritance problem. Lexicographic (lex) entailment [2, 7] (section 2.3) which is justified by presumptions of typicality, independence, priority and specificity, and maximum entropy (me) entailment [4, ?] (section 3) which uses the principle of maximum entropy as a means of selecting the least biased probability distribution associated with an incomplete set of probabilistic constraints. Both systems are described and shown to exhibit the required behaviour.

It is shown (section 4) that it is possible to recreate the lexicographic closure of a set of defaults under maximum entropy by assigning appropriate strengths to the defaults. An algorithmic definition is given which translates the lex-ordering into an me-ranking and hence finds a set of canonical me-strengths for the defaults. This implies that lex-entailment can be thought of as a subset of me-entailment corresponding a particular choice of strength assignments.

The dynamic behaviour of the system of lex-entailment is examined (section 5). It is shown that the semantics of a default, when interpreted as its canonical me-strength, is highly dependent on its surrounding defaults with respect to the lex-ordering. Under maximum entropy, however, a default's semantics can be fixed and independent of other defaults. This finding is used to argue that the lex-ordering requires the user to accept some rather strong assumptions.

By connecting the two systems, the intuitions underlying lex-entailment are clarified, and, it is argued, the more general approach of me-entailment is both more expressive, since it allows variable strength defaults to be represented explicitly, and more justifiable, by virtue of its grounding in a well-understood principle of reasoning rationally from incomplete information.

2 Lexicographic entailment

2.1 Definitions and notation

First some preliminary definitions and notation. A finite propositional language $\mathcal{L}$ is made up of propositions a, b, c, ... and the usual connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. A *default* is a pair of propositions or formulas joined by a default connective $\Rightarrow$, e.g., $a \Rightarrow b$. The language has a finite set of models, $\mathcal{M}$. A model m *verifies* a default $a \Rightarrow b$ if $m \models a \wedge b$, where $\models$ is classical entailment, and *falsifies* it if $m \models a \wedge \neg b$. A default r *tolerates* a set of defaults Δ iff it has a verifying model which does not falsify any defaults in Δ; such a model will be called a *confirming* model of r with respect to Δ.

It has been shown in [8] that any consequence relation that satisfies all the rules of System P plus that of rational monotonicity is equivalent to a total ordering of the models of $\mathcal{M}$ and, conversely, any total ordering of the models of $\mathcal{M}$ is equivalent to a so-called rational consequence relation. The rank of a formula in such an ordering is the rank of its minimal satisfying model(s). A ranking, κ, is called *admissible* with respect to a set of defaults, Δ, iff for all $a \Rightarrow b \in \Delta$, $\kappa(a \wedge b) \prec \kappa(a \wedge \neg b)$. Similarly, a default $c \Rightarrow d$ belongs to the rational consequence relation determined by κ iff $\kappa(c \wedge d) \prec \kappa(c \wedge \neg d)$. Three mechanisms for generating such a total order are provided by System Z (section 2.2), the lex-ordering (section 2.3) and the me-ranking (section 3).

2.2 System Z

System Z [10], or equivalently rational closure [8], can be defined as follows. Given a p-consistent set of defaults[1], Δ, it is possible to identify a subset Δ_0 made up of all the defaults which tolerate all other defaults in Δ. Then, given $\Delta - \Delta_0$ it is possible to identify another subset, Δ_1, made up of all the defaults which tolerate all members of $\Delta - \Delta_0$, and the process continues until all the remaining defaults tolerate each other. This process gives the unique *z-partition* $\Delta = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$. Each default is assigned a *z-rank* which is the index of the Δ_i to which it belongs, and each model is assigned a z-rank of 1 plus

[1] A set of defaults is p-consistent iff every non-empty subset is confirmable [1] or, equivalently, iff there exists an admissible ranking function with respect to that set.

m	b	f	p	w	z
m_1	0	0	0	0	0
m_2	0	0	0	1	0
m_3	0	0	1	0	2
m_4	0	0	1	1	2

m	b	f	p	w	z
m_5	0	1	0	0	0
m_6	0	1	0	1	0
m_7	0	1	1	0	2
m_8	0	1	1	1	2

m	b	f	p	w	z
m_9	1	0	0	0	1
m_{10}	1	0	0	1	1
m_{11}	1	0	1	0	1
m_{12}	1	0	1	1	1

m	b	f	p	w	z
m_{13}	1	1	0	0	1
m_{14}	1	1	0	1	0
m_{15}	1	1	1	0	2
m_{16}	1	1	1	1	2

Fig. 1. The z-rankings for the penguin example.

the highest z-rank of all the defaults it falsifies, or 0 if it falsifies no defaults. This z-ranking is admissible with respect to Δ and z-entailment is determined from this ranking. Since the higher the z-rank of a model the more abnormal (in the sense of being less probable) it is, a default is z-entailed iff the z-rank of its minimal verifying model(s) is strictly less than the z-rank of its minimal falsifying model(s) (meaning that it is more normal for the default to be verified than falsified).

Example 1 (Penguins).

$$\Delta = \{b \Rightarrow f, b \Rightarrow w, p \Rightarrow b, p \Rightarrow \neg f\}$$

(the intended interpretation of this database is that birds fly, birds have wings, penguins are birds but penguins do not fly). The z-partition of this database is:

$$\Delta_0 = \{b \Rightarrow f, b \Rightarrow w\} \qquad \text{and} \qquad \Delta_1 = \{p \Rightarrow b, p \Rightarrow \neg f\}$$

Here $\mathcal{L}$ has four atoms so $\mathcal{M}$ contains only 16 models. Figure 1 enumerates these models along with their z-ranks. To establish whether the default "penguins have wings" is z-entailed, it is necessary to consider the z-ranks of the minimal verifying and falsifying models of $p \Rightarrow w$ (m_{12} and m_{11}, respectively):

$$\mathsf{z}(p \wedge w) \quad = \quad 1 \quad = \quad \mathsf{z}(p \wedge \neg w)$$

and so $p \Rightarrow w$ is not z-entailed. •

This example illustrates one of the problems with z-entailment—it does not allow inheritance to exceptional subclasses.

2.3 The lexicographic ordering

The lexicographic ordering was proposed by Lehmann [7] who argued that the behaviour of the ideal rational consequence relation should satisfy four presumptions of typicality, independence, priority and specificity. He also drew attention to the differences between the presumptive reading of a default, as first developed by Reiter [11], and the prototypical reading for which, he claims, the rational closure [8, 10] is the "correct formalization". A more flexible variant of Lehmann's lexicographic closure is given by Benferhat *et al.* [2] who allow the

m	b f p w	lex
m_1	0 0 0 0	(0,0)
m_2	0 0 0 1	(0,0)
m_3	0 0 1 0	(0,1)
m_4	0 0 1 1	(0,1)

m	b f p w	lex
m_5	0 1 0 0	(0,0)
m_6	0 1 0 1	(0,0)
m_7	0 1 1 0	(0,2)
m_8	0 1 1 1	(0,2)

m	b f p w	lex
m_9	1 0 0 0	(2,0)
m_{10}	1 0 0 1	(1,0)
m_{11}	1 0 1 0	(2,0)
m_{12}	1 0 1 1	(1,0)

m	b f p w	lex
m_{13}	1 1 0 0	(1,0)
m_{14}	1 1 0 1	(0,0)
m_{15}	1 1 1 0	(1,1)
m_{16}	1 1 1 1	(0,1)

Fig. 2. The lex-tuples for the penguin example.

user to determine the priorities of defaults, rather than being restricted to the ranks determined by the z-partition.

Lexicographic entailment is defined as follows. The lex-ordering over the models of $\mathcal{L}$ is based on the z-partition but takes into account all defaults violated by a model, not just that with the greatest z-rank. The result is a form of entailment which is a direct extension of System Z in the sense that all z-entailed defaults are also lex-entailed.

Given a set of defaults, Δ, and its z-partition, $\Delta_0 \cup \Delta_1 \ldots \cup \Delta_n$, each model is assigned an $(n+1)$-tuple with the number of defaults violated in partition-set Δ_i appearing in position i of the tuple. The lex-ordering of tuples (and hence models) is determined by considering the last elements of the tuples first. If one tuple has fewer default violations in the highest tuple element, it is lower (or preferred) in the lex-ordering; otherwise the next highest tuple element is considered. For example, $(1,1,0) \prec (0,0,2)$ and $(2,0,1) \prec (0,1,1)$. From the lex-ordering, entailment is determined as usual by comparing the lex-tuples of the minimal verifying and falsifying models of a default.

Example 2 (Penguins (continued)). Figure 2 gives the lex-tuples of default violations for each model. Comparing the minimal verifying and falsifying models of $p \Rightarrow w$ gives:

$$\mathtt{lex}(p \wedge w) \quad = \quad (1,0) \quad \prec \quad (2,0) \quad = \quad \mathtt{lex}(p \wedge \neg w)$$

and so $p \Rightarrow w$ is lex-entailed. •

As the example demonstrates, lex-entailment does provide for inheritance to exceptional subclasses.

3 Maximum entropy entailment

Ranking functions can be viewed as an abstraction of a probabilistic semantics for defaults [10]. A default can be thought of as a constraint on a probabilitity distribution (PD) and so a set of defaults constrains the possible PDs. Usually these will not be sufficient to completely specify a single PD. Goldszmidt *et al.* [4] developed the maximum entropy approach to default reasoning by applying the principle of maximum entropy which is a well understood means of selecting that PD which satisfies a set of constraints and contains the least extra information

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me-algorithm

Input: a set of variable strength defaults, {r_i : a_i =s_i=> b_i}.
Output: an me-valid ranking, κ, if one exists.

[1] Initialise all κ(r_i) = INF.
[2] While any κ(r_i) = INF do:
   (a) For all r_i with κ(r_i) = INF, compute
       MINV(r_i) + s_i.
   (b) For all such r_i with minimal MINV(r_i) + s_i,
       compute MINF(r_i).
   (c) Select r_j with minimal MINF(r_i).
   (d) If MINF(r_j) = INF let κ(r_j) := 0
       else let κ(r_j) := s_j + MINV(r_j) − MINF(r_j).
[3] Assign ranks to models using equation (2).
[4] Check constraints (1) to verify this is an me-valid ranking.
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Fig. 3. The me-algorithm

[5]. If one has to select a PD from all possible ones, choosing one other than that which has maximum entropy means making additional assumptions or implicitly assuming extra constraints.

It would be useful therefore to be able to compare systems of default reasoning with the answers obtained from the me-approach in order to understand what implicit assumptions underlie those systems. In order to do this, the me-approach originally proposed by Goldszmidt *et al.* [4] has been extended by Bourne and Parsons [3] to admit arbitrary sets of defaults with variable strengths. The me-ranking of a set of defaults $\{r_i\}$ with strengths $\{s_i\}$ can be found by applying the me-algorithm given in figure 3. The me-algorithm looks for a solution to the following set of non-linear simultaneous equations:

$$\min_{m \models a_i \wedge \neg b_i} [\mathtt{me}(m)] \quad = \quad s_i + \min_{m \models a_i \wedge b_i} [\mathtt{me}(m)] \tag{1}$$

$$\mathtt{me}(m) = \sum_{\substack{r_i \\ m \models a_i \wedge \neg b_i}} \mathtt{me}(r_i) \tag{2}$$

The solution is a set of me-ranks corresponding to each default, $\{\mathtt{me}(r_i)\}$. From these, using (2), the me-ranks of each model, $\{\mathtt{me}(m)\}$, can be determined.

As discussed in detail in [3], the ranking found by the me-algorithm may not always be a unique solution to the equations, indeed for certain strength assignments no solution may exist, however the algorithm does find the unique solution when there is one.

Example 3 (Penguins (continued)). Let each rule r_i have an associated strength of s_i. The constraint equations (1) give rise to:

$\mathtt{me}(r_1) = s_1$ $\qquad$ $\mathtt{me}(r_3) = s_3 + \min(\mathtt{me}(r_1), \mathtt{me}(r_2))$

$\mathtt{me}(r_2) = s_2 + \min(\mathtt{me}(r_1), \mathtt{me}(r_3))$ $\qquad$ $\mathtt{me}(r_4) = s_4$

m	$b\ f\ p\ w$	me	m	$b\ f\ p\ w$	me
m_1	0 0 0 0	0	m_9	1 0 0 0	s_1+s_4
m_2	0 0 0 1	0	m_{10}	1 0 0 1	s_1
m_3	0 0 1 0	s_1+s_2	m_{11}	1 0 1 0	s_1+s_4
m_4	0 0 1 1	s_1+s_2	m_{12}	1 0 1 1	s_1
m_5	0 1 0 0	0	m_{13}	1 1 0 0	s_4
m_6	0 1 0 1	0	m_{14}	1 1 0 1	0
m_7	0 1 1 0	$2s_1+s_2+s_3$	m_{15}	1 1 1 0	$s_1+s_3+s_4$
m_8	0 1 1 1	$2s_1+s_2+s_3$	m_{16}	1 1 1 1	s_1+s_3

Fig. 4. The me-ranks for the penguin example.

which have the unique solution $\mathtt{me}(r_1)=s_1$, $\mathtt{me}(r_2)=s_1+s_2$, $\mathtt{me}(r_3)=s_1+s_3$, and $\mathtt{me}(r_4)=s_4$. The me-rankings are given in figure 4.

Comparing the minimal verifying and falsifying models of $p \Rightarrow w$ gives:

$$\mathtt{me}(p \wedge w) \quad = \quad s_1 \quad < \quad s_1+\min(s_2,s_4) \quad = \quad \mathtt{me}(p \wedge \neg w)$$

and so $p \Rightarrow w$ is me-entailed.

Clearly, this default is me-entailed under any strength assignment because the solution for the $\{\mathtt{me}(r_i)\}$ holds for any $\{s_i\}$. This will not be true in general as different strength assignments may map to qualitatively different me-rankings.•

As the example demonstrates, me-entailment also provides for inheritance to exceptional subclasses.

4 Translating lexicographic to maximum entropy

By changing the strengths assigned to defaults, it is possible to produce many different me-rankings, all of which represent rational consequence relations [3]. The me-rankings differ because the different strengths change the default information being encoded. However, the me-ranking corresponding to any given set of strengths represents the least biased estimate of the underlying probability distribution [5]. In contrast, the lex-ordering is unique and fixed for a given set of defaults [7]. It follows that the lex-ordering implies some additional assumptions are being made about what default information represents and it is reasonable to ask what these might be. By showing that the lex-ordering can be equated to a class of me-rankings, this section aims to make explicit the underlying semantics of lexicographic entailment.

The similarity between these two forms of entailment lies in the fact that in both methods the ordering makes use of all defaults falsified by each model. In the lex-ordering the tuple represents the position and number of defaults falsified, whilst for the me-ranking, the me-rank of each model is the sum of the me-ranks of each default it falsifies. Thus by assigning appropriate me-ranks to the defaults it is possible to create an me-ranking which is equivalent to the lex-ordering, in the sense that the ordering of models is the same. It is then possible

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Translation algorithm

Input: A partitioning of Δ, Δ_0 ∪ Δ_1 ... ∪ Δ_n.
Output: The canonical me-ranking, me_Δ, plus associated strength
assignment, {s_i}.

[1] Let me(r_i) = 1 for all r_i ∈ Δ_0.
[2] For k = 1 to n:
    (a) Let me(Δ_k) = (|Δ_{k-1}| + 1) * me(Δ_{k-1}).
    (b) Let me(r_i) = me(Δ_k) for all r_i ∈ Δ_k.
[3] For each r_i:
    (a) Find the ranks of its minimal verifying and falsifying models,
        me_Δ(v_{r_i}) and me_Δ(f_{r_i}), using equation (2).
    (b) Set s_i = me_Δ(f_{r_i}) - me_Δ(v_{r_i}).
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Fig. 5. The translation algorithm

to compute what strength assignment over defaults gives rise to this me-ranking. From the characteristics of this strength assignment, it is possible to interpret what exactly the lex-ordering means in terms of what the implications are for the relative strengths of defaults.

In order to create an me-ranking equivalent to the lex-ordering, all defaults in a given partition-set should have the same me-rank. This ensures that whenever two models falsify different defaults which belong to the same partition-set, the "penalty" associated with each is the same. In addition, it must always be worse to falsify defaults in a certain partition-set than to falsify *any number* of defaults in lower sets. Thus the me-rank assigned to defaults in the partition-set Δ_i, denoted $\mathtt{me}(\Delta_i)$, must be greater than the sum of the me-ranks of all defaults in lower sets. The translation-algorithm given in figure 5 accomplishes such an assignment of me-ranks to defaults.

Note that the me-rank assignment in step [2](a), is arbitrary to the extent that any integer greater than the sum of the me-ranks of all defaults in lower partition sets would suffice. Thus there is a whole class of me-rankings which are equivalent to a given lex-ordering.

Once the me-ranks have been assigned to rules it is a simple matter to calculate the corresponding strength assignment necessary to achieve this me-ranking: each default has a strength which is equivalent to the difference between the me-ranks of its minimal falsifying and verifying models. The strength of any default in the me-ranking found using the translation algorithm will be called the *canonical me-strength* of that default. Note that not only the defaults in the original set, but also any default which is lex-entailed (and hence me-entailed in the canonical me-ranking) will have an associated canonical me-strength[2].

[2] In [3], the me-ranking is shown to be the unique solution to equations (1) and (2) if it satisfies a condition termed "robustness". If the lex-ordering is robust then so is the canonical me-ranking which in turn implies that the canonical me-strength

The following example shows the translation algorithm at work leading to a canonical me-strength assignment which gives an identical rational consequence relation to that given by the lex-ordering.

Example 4 (Bears).

$$\Delta = \{r_1 : b \Rightarrow d, r_2 : t \Rightarrow b, r_3 : t \Rightarrow \neg d, r_4 : b \Rightarrow h, r_5 : t \wedge l \Rightarrow d\}$$

(the intended interpretation of this knowledge base is that bears are dangerous, teddies are bears, teddies are not dangerous, bears like honey, and teddies with loose glass eyes are dangerous). The z-partition has three partition-sets:

$$\Delta_0 = \{b \Rightarrow d, b \Rightarrow h\} \qquad \Delta_1 = \{t \Rightarrow \neg d, t \Rightarrow b\} \qquad \Delta_2 = \{t \wedge l \Rightarrow d\}$$

Following the algorithm, set $\mathtt{me}(r_1) = \mathtt{me}(r_4) = 1$; then $\mathtt{me}(\Delta_1) = 3$, so $\mathtt{me}(r_2) = \mathtt{me}(r_3) = 3$; finally $\mathtt{me}(\Delta_2) = 9$, so $\mathtt{me}(r_5) = 9$. This me-ranking is robust and corresponds to a strength assignment of $(1, 2, 2, 1, 7)$. The lex-ordering and canonical me-ranking both induce the same rational consequence relation. Consider the default "teddies which are dangerous and do not like honey are bears". To see whether this is entailed, it is necessary to examine the minimal verifying and falsifying models of $t \wedge d \wedge \neg h \Rightarrow b$:

$$\mathtt{lex}(t \wedge d \wedge \neg h \wedge b) = (1,1,0) \quad \prec \quad \mathtt{lex}(t \wedge d \wedge \neg h \wedge \neg b) = (0,2,0)$$
$$\mathtt{me}_\Delta(t \wedge d \wedge \neg h \wedge b) = 4 \quad < \quad \mathtt{me}_\Delta(t \wedge d \wedge \neg h \wedge \neg b) = 6$$

and so this default is both lex-entailed and canonically me-entailed. •

The translation algorithm finds a set of canonical me-strengths for any set of defaults that leads to an me-consequence relation which coincides with the lex-consequence relation. In fact there is an infinite class of such strength assignments. The implication is that the lex-consequence relation is just a special case of the me-consequence relation. So what are the additional assumptions underlying the lex-ordering?

The canonical me-strengths of defaults increase exponentially with the index of the partition set to which they belong. Effectively, the defaults in higher sets are deemed to hold more strongly under lexicographical entailment. Now, the z-ranking actually represents the exponent of qualitative probabilities, or the relative order of magnitude of models. The strength of a default, in contrast, represents an order of magnitude relation between sets of models. When the lex-ordering is translated into an me-ranking, the strength associated with each default is inversely connected with the probability of its minimal verifying model so that the strength of a default increases as the probability of it actually being

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assignment leads to a unique me-ranking. For non-robust lex-orderings, the canonical me-strength assignment might lead to multiple me-solutions. However, since the canonical me-ranking is already arbitrary to some extent, this does not have a bearing on the analysis and can be safely ignored. Readers interested in robustness and multiple solutions are referred to [3].

verified decreases. The principles used by Lehmann to justify the lex-ordering [7] bear no relation to this observation, however. Benferhat's version of the lex-ordering [2], which allows the user to specify the priorities explicitly, has a better justification since at least then the increase in strength can be viewed as the realisation of the default priorities which the user has chosen to impose.

However, in both lex-systems, the canonical me-strengths which the translation algorithm supplies do not directly correspond either to the partition sets or to the priorities the user assigns. This is because for two defaults which have the same priority, the lex-tuples of their minimal verifying and falsifying models may differ slightly leading to differences in their canonical me-ranks. In both systems, the priorities only determine the order of magnitude of the canonical me-strengths which may vary slightly for defaults of the same priority. So although the lex-ordering allows the priorities to be specified, this cannot be achieved in isolation from the other defaults. In contrast, using the me-approach directly allows the user to specify the default priorities explicitly and independently. Thus, if the object of using a lexicographic ordering is to allow the knowledge engineer to make explicit his judgments about default priorities, it can be argued that using maximum entropy and variable strengths is the fairest and most transparent way to achieve this.

5 Behaviour of lexicographic entailment

It is interesting to examine the behaviour of systems of default reasoning from the meta-level perspective. For example, it is well known that while System P maps a set of defaults into a nonmonotonic consequence relation, System P itself is strictly monotonic on the addition of further defaults. This behaviour has been termed "semi-monotonic" by Pearl [10] but, in fact, System P behaves classically if defaults are given the appropriate semantics (e.g., let a default correspond to the set of its admissible ranking functions).

The behaviour of systems when their consequences are learned, i.e., an entailed default is added to the set which entailed it, can be used to argue for the reasonableness of adopting such a system. It has been suggested [9] that systems should satisfy rules like those of System P at the meta-level although how these should be interpreted is not always obvious. Lehmann himself pointed out that lex-entailment does not satisfy cautious monotonicity since adding entailed defaults may lead to the retraction of previous conclusions [7]. The following theorems make clear why this occurs and what the implications are for the canonical me-strengths with which defaults are entailed.

Theorem 5 shows that, provided a default is not entirely unexpected, i.e., its converse is not z-entailed, then the z-partition (and hence the z-rank of defaults) will not change radically on the addition of that default. In fact, a small ripple effect occurs with the new default being added to the appropriate partition set and defaults of equal or higher rank may or may not be 'shunted up' by one degree.

Theorem 5 (Dynamics of z-partition). *Consider a set of defaults, Δ, with z-partition $\Delta_0 \cup \ldots \cup \Delta_n$. Let r be a default such that the z-rank, k, of its minimal verifying model is not more than the z-rank of its minimal falsifying model (equivalently, the converse of r is not z-entailed by Δ). Then (1) the z-partition of $\Delta' = \{r\} \cup \Delta$ is such that $\Delta'_i = \Delta_i$ for $i < k$, (2) $r \in \Delta'_k$ and (3) for all $r' \in \Delta_{j \geq k}$, either $r' \in \Delta'_j$ or $r' \in \Delta'_{j+1}$.*

Proof. All confirming models for the defaults in $\Delta_0 \cup \ldots \cup \Delta_{k-1}$ neither verify nor falsify r by the conditions of the theorem, hence the first k partition-sets in the new z-partition will be the same, that is, for $i < k$, $\Delta'_i = \Delta_i$, as required.

Now if v_r is a minimum verifying model of r, it is also a confirming model for r wrt $\{r\} \cup \Delta_k \cup \ldots \cup \Delta_n$, since it may falsify defaults in $\Delta_{i<k}$ but not in higher sets. Thus $r \in \Delta'_k$, as required.

Finally, consider $v_{r'}$, a verifying model for some default $r' \in \Delta_k$ which previously confirmed r' wrt $\Delta_k \cup \ldots \cup \Delta_n$. If $v_{r'}$ satisfies r then it is also a confirming model of r' wrt $\{r\} \cup \Delta_k \cup \ldots \cup \Delta_n$, so $r' \in \Delta'_k$. Otherwise r' does not tolerate $\{r\} \cup \Delta_k \cup \ldots \cup \Delta_n$. Therefore separate Δ_k into those defaults which tolerate $\{r\} \cup \Delta_k \cup \ldots \cup \Delta_n$, say Δ_{T_k}, and those which do not, say $\Delta_{\neg T_k}$. Then $\Delta'_k = \{r\} \cup \Delta_{T_k}$ and it remains to partition $\Delta_{\neg T_k} \cup \Delta_{k+1} \ldots \cup \Delta_n$. Clearly all defaults in $\Delta_{\neg T_k}$ tolerate $\Delta_{\neg T_k} \cup \Delta_{k+1} \ldots \cup \Delta_n$ since they did previously and so $\Delta_{\neg T_k} \subset \Delta'_{k+1}$. Separate Δ_{k+1} into those defaults which tolerate $\Delta_{\neg T_k} \cup \Delta_{k+1} \ldots \cup \Delta_n$, say $\Delta_{T_{k+1}}$, and those which do not, say $\Delta_{\neg T_{k+1}}$. Then $\Delta'_{k+1} = \Delta_{\neg T_k} \cup \Delta_{T_{k+1}}$ and it remains to partition $\Delta_{\neg T_{k+1}} \cup \Delta_{k+2} \ldots \cup \Delta_n$. Proceeding in this way, the z-partition of Δ' is formed such that for any default, $r' \in \Delta_{j \geq k}$, it holds that either $r' \in \Delta'_j$ or $r' \in \Delta'_{j+1}$, as required. •

Theorem 6 shows that if a default is entirely expected, i.e., is z-entailed, then no z-ranks change. This demonstrates why System Z can be called the rational *closure* of the set, since the addition of a z-entailed default will not lead to any new z-conclusions; there will undoubtedly be further lex-conclusions, however.

Theorem 6. *Given the conditions of theorem 5, if the z-rank, k, of the minimal verifying model of r is strictly less than the z-rank of its minimal falsifying model according to Δ (equivalently, r is z-entailed by Δ), then $\Delta'_k = \{r\} \cup \Delta_k$ and $\Delta'_i = \Delta_i$ for $i \neq k$.*

Proof. Since r is z-entailed by Δ, all confirming models of defaults in Δ_k have z-rank k and therefore cannot be falsifying models of r. Hence all defaults in Δ_k tolerate $\{r\} \cup \Delta_k \cup \ldots \cup \Delta_n$, and $\Delta'_k = \{r\} \cup \Delta_k$. All other partition-sets remain unchanged. •

Finally, theorem 7 demonstrates that adding a default to a set which lex-entailed it, leads to the default obtaining a higher canonical me-strength. Clearly, this is to be expected since when the lex-entailed default is learnt, violating it takes on more significance.

Theorem 7. *If using $\mathtt{me}_\Delta$, the canonical me-ranking for Δ, r is me-entailed with strength s, then using $\mathtt{me}_{\Delta'}$, the canonical me-ranking for $\Delta' = \{r\} \cup \Delta$, r is me-entailed with strength $s' > s$.*

Proof. Let the z-partition of Δ be $\Delta_0 \cup \ldots \cup \Delta_n$ and the lex-equivalent me-ranks associated with each partition set be $\mathtt{me}(\Delta_0), \ldots, \mathtt{me}(\Delta_n)$. Let the minimal verifying and falsifying models for r in the me-ranking be v_r and f_r, respectively. Then $s = \mathtt{me}_\Delta(f_r) - \mathtt{me}_\Delta(v_r)$.

First suppose that r is z-entailed by Δ so that if $\mathtt{z}(v_r) = k$ then $\mathtt{z}(f_r) > k$. Then by theorem 6 the z-partition of Δ' has $\Delta'_k = \{r\} \cup \Delta_k$ and $\Delta'_i = \Delta_i$ for $i \neq k$. Hence $\mathtt{me}(\Delta'_i) = \mathtt{me}(\Delta_i)$ for $i \leq k$ and $\mathtt{me}(\Delta'_j) > \mathtt{me}(\Delta_j)$ for $j > k$. Now, $\mathtt{me}_{\Delta'}(v_r) = \mathtt{me}_\Delta(v_r)$ since v_r only falsifies defaults in partition-sets Δ_0 to Δ_{k-1}. However, f_r now falsifies an extra default, r itself, and so its me-rank must be higher by at least $\mathtt{me}(\Delta_k)$. Hence $s' = \mathtt{me}_{\Delta'}(f_r) - \mathtt{me}_{\Delta'}(v_r) \geq \mathtt{me}_\Delta(f_r) + \mathtt{me}(\Delta_k) - \mathtt{me}_\Delta(v_r) > s$, as required.

Now suppose that r is only lex-entailed so that $\mathtt{z}(v_r) = \mathtt{z}(f_r) = k$. Then the z-partition of Δ' is as described in theorem 5 so that $r \in \Delta'_k$. Now $\mathtt{me}(\Delta'_i) = \mathtt{me}(\Delta_i)$ for $i \leq k$. Again $\mathtt{me}_{\Delta'}(v_r) = \mathtt{me}_\Delta(v_r)$. However, since $\mathtt{z}(f_r) = k$ it follows that $\mathtt{me}_\Delta(f_r) < \mathtt{me}(\Delta_k)$ but $\mathtt{me}_{\Delta'}(f_r) \geq \mathtt{me}(\Delta_k)$. Hence $s' = \mathtt{me}_{\Delta'}(f'_r) - \mathtt{me}_{\Delta'}(v_r) \geq \mathtt{me}(\Delta_k) - \mathtt{me}_\Delta(v_r) > \mathtt{me}_\Delta(f_r) - \mathtt{me}_\Delta(v_r) = s$, as required. •

Theorem 7 shows that adding a default to a set which lex-entailed it leads to it obtaining a higher canonical me-strength than that with which it was previously me-entailed. This would seem to be an explanation of the fact that lex-entailment fails to satisfy cautious monotonicity. Syntactically, theorem 5 confirms this since the addition of a lex-entailed default may lead to a revised z-partition which no longer lex-entails old conclusions. However, one could argue that, according to the semantic interpretation of lex-entailment as a form of me-entailment, it is not *possible* to add a lex-entailed default to a set without changing its semantics, i.e., its canonical me-strength. In a sense, this argument implies that cautious monotonicity is simply not applicable to lex-entailment since the semantics of a default cannot be specified independently of its surrounding defaults.

The behaviour of me-entailment on the addition of me-entailed defaults is interesting. It depends critically on the strength assigned to the given default compared with the degree to which it is me-entailed[3]. If it is assigned a lower strength then no admissible me-ranking exists, whilst if it is assigned a higher strength a revised unique me-ranking is produced. If the added default is assigned a strength equal to the degree to which it was previously entailed, it is usually the case that there are multiple solutions for the me-ranking. An me-ranking with the added default taking zero me-rank is one solution—one could say in this case that the default is redundant—but there may be other solutions in which it is not the added default which is redundant but one of the originals. A more detailed account of these findings may be found in [3]. Thus it is possible for the addition of the default to lead to the same me-ranking, that is, me-entailment does satisfy cautious monotonicity, however one must be careful since this solution may not be unique.

[3] That is, the difference between the me-ranks of its minimal falsifying and verifying models.

6 Conclusion

This paper has compared lexicographic entailment with maximum entropy entailment and found the former to be a special case of the latter. It has been argued that the me-approach is better justified since it is based on a well-understood principle of indifference [5], and that it is a better method for representing judgments about the relative priorities between defaults because these can be made explicitly and independently. The behaviour of both systems was also examined to show why lexicographic entailment fails to satisfy the meta-rule of cautious monotonicity and how maximum entropy entailment does satisfy it under certain conditions and with certain caveats.

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