Proof theoretic reasoning in System P

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Abstract

This paper investigates how the rules of System P might be used in order to construct proofs for default rules which take into account the bounds on the probabilities of the consequents of the defaults. The paper defines a proof system, shows that it is sound, and then discusses at length the completeness of the system, and the kind of proofs that it can generate.

Introduction

Default reasoning has been widely studied in artificial intelligence. One of the the most influential piece of work within this area is that of Kraus *et al.* (1990). Kraus *et al.* investigated the properties of different sets of Gentzen-style proof rules for non-monotonic consequence relations, and related these sets of rules to the model-theoretic properties of the associated logics. Their major result was that a particular set of proof rules generated the same set of consequences as a logic in which there is a preference order over models. This system of proof rules was termed System P by Kraus *et al.*, the P standing for "preferential". System P has been the subject of much research, and is now widely accepted as the weakest interesting non-monotonic system; it sanctions the smallest acceptable set of conclusions from a set of default statements.

The reason that we are interested in the rules of System P is that, in addition to a semantics in terms of a preference order over models, they also have a semantics in terms of infinitesimal probabilities (Adams 1975; Pearl 1988). This semantics can be extended to deal with finite probabilities (Bourne & Parsons 1998), an extension which promises to make System P more useful in practice. However, the use of finite probabilities raises a new problem. In order to keep track of the probability attached to each default, it is necessary to establish a new mechanism for inferring new defaults along with their probabilities, and this is the subject of this paper.

Entailment in System P

System P is concerned with conditional assertions of the form $\alpha \mid\!\sim \beta$. In this context α and β are well-formed formulae of classical propositional logic, and $\mid\!\sim$ is a binary relation between pairs of formulae. The probabilistic semantics for System P assumes that the propositional variables are the basis of a joint probability distribution which is constrained by the assertions. Each assertion states that the conditional probability of its consequent given its antecedent is greater than or equal to $1-\epsilon$ for any $\epsilon > 0$. Thus:

Definition 1 *The conditional assertion* $\alpha \mid\!\sim \beta$ *denotes the fact that* $\Pr(\beta|\alpha) \geq 1-\epsilon$ *for all* $\epsilon > 0$.

Probabilistic consistency is defined as the existence of at least one probability distribution which satisfies these constraints (Adams 1975); probabilistic entailment of a further conditional is defined as probabilistic inconsistency of its counterpart, that is:

Definition 2 $\alpha \mid\!\sim \beta$ *is* p-entailed *by* Δ *iff* $\Delta \cup \{\alpha \mid\!\sim \neg\beta\}$ *is not p-consistent.*

This implies that all probability distributions that satisfy Δ also satisfy $\alpha \mid\!\sim \beta$. However this result may only be achieved by using infinitesimal analysis so that the derived conditional will be constrained to be greater than $1-\delta$ for any $\delta > 0$ if the ϵ of the original conditionals is made small enough. This can be paraphrased as saying that System P allows us to make our conclusions as close to certainty as we like, provided the conditional probabilities associated with the initial assertions are sufficiently close to certainty.

Using this interpretation of the rules thus means assuming that we are able to give the conditional assertions arbitrarily high conditional probabilities. Often we have less reliable information, and so ϵ is not infinitesimal. When such a set of conditional assertions are used to derive new assertions and these new assertions are themselves used as the basis for new deductions, then some ϵ values will be far from infinitesimal and it is not possible to assess the impact of these non-infinitesimal values on the strength of the new assertions.

Because of this concern, we investigated (Bourne & Parsons 1998) the impact of non-infinitesimal ϵ values

$\alpha \mid\!\sim_0 \alpha$	Reflexivity	$\dfrac{\models \alpha \leftrightarrow \beta, \alpha \mid\!\sim_{\epsilon_1} \gamma}{\beta \mid\!\sim_{\epsilon_1} \gamma}$	Left Logical Equivalence
$\dfrac{\models \beta \rightarrow \gamma, \alpha \mid\!\sim_{\epsilon_1} \beta}{\alpha \mid\!\sim_{\epsilon_1} \gamma}$	Right Weakening	$\dfrac{\alpha \mid\!\sim_{\epsilon_1} \beta, \alpha \mid\!\sim_{\epsilon_2} \gamma}{\alpha \mid\!\sim_{\epsilon_1+\epsilon_2} \beta \wedge \gamma}$	And
$\dfrac{\alpha \mid\!\sim_{\epsilon_1} \beta, \alpha \mid\!\sim_{\epsilon_2} \gamma}{\alpha \wedge \beta \mid\!\sim_{\frac{\epsilon_2}{1-\epsilon_1}} \gamma}$	Cautious Monotonicity	$\dfrac{\alpha \mid\!\sim_{\epsilon_1} \gamma, \beta \mid\!\sim_{\epsilon_2} \gamma}{\alpha \vee \beta \mid\!\sim_{\epsilon_1+\epsilon_2} \gamma}$	Or

Figure 1: Rules with associated bounds

in the context of a set of rules of inference for System P given by Kraus *et al*, showing how ϵ values are propagated. The results of this investigation are the rules in Figure 1. The rules are written in the usual Gentzen style, with antecedents above the line and consequents below it. Thus the rule 'And' says that if it is possible to derive $\alpha \mid\!\sim \beta$ such that $\Pr(\beta \mid \alpha) \geq 1-\epsilon_1$ and $\alpha \mid\!\sim \gamma$ such that $\Pr(\gamma \mid \alpha) \geq 1-\epsilon_2$, then it is possible to derive $\alpha \mid\!\sim \beta \wedge \gamma$ such that $\Pr(\beta \wedge \gamma \mid \alpha) \geq 1-(\epsilon_1+\epsilon_2)$.

While this work solved the problem of determining the impact of the non-infinitesimal values, it fell short of providing a practical reasoning system. The problem is that although in System P we can tell whether or not $\alpha \mid\!\sim \beta$ follows from the initial set of defaults, the procedures for determining this do not permit the propagation of the ϵ values. Thus we can tell if $\alpha \mid\!\sim \beta$ follows, and so we can find out if a proof exists, but we can't determine the associated ϵ value. What we need is a proof theory which allows the ϵ values to be propagated through the proof so that every inferred default has its ϵ value determined, and providing such a proof theory is the subject of this paper.

A proof theory for System P

Normally in generating a proof theory for some logical system the procedure (Gabbay 1996) is to establish two rules for each connective in the underlying language. One rule relates to introducing the connective into a formula, and one relates to eliminating the connective from a formula. The set of rules then define all the legal transformations between formulae, and thus define what may be proved from some initial set of formulae. The process of defining a proof theory thus proceeds from the underlying language to the proof rules.

The situation here is a little different. System P already has a set of proof rules defined. However, these rules do not include introduction and elimination rules for all the connectives in the underlying language, and so do not support a conventional proof theory. However, it is possible to use the existing rules to define a proof theory for a significant part of the underlying language of System P, and this is the approach we adopt.

The proof system

We start with a set of propositions $\mathcal{S}$, a set of connectives, $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Rightarrow\}$, and the following rules for building well-formed formulae in this language:

1. If $\alpha \in \mathcal{S}$, then α is a basic well-formed formula (*bwff*).
2. If α and β are *bwff*s then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\alpha \leftrightarrow \beta$ are *bwff*s.
3. If γ and δ are *bwff*s, then $\gamma \Rightarrow_\epsilon \delta$ is a default well-formed formula (*dwff*).
4. Nothing else is a *bwff* or a *dwff*.

Together all these formulae constitute a language $\mathcal{L}_\mathcal{S}$. The denotation of basic well-formed formulae is as in propositional logic, while the meaning of *dwff*s is the following:

Definition 3 *The default* $\gamma \Rightarrow_\epsilon \delta$ *is taken to mean* $\Pr(\delta \mid \gamma) \geq 1-\epsilon$.

Comparing Definitions 1 and 3 it is clear that the defaults of $\mathcal{L}_\mathcal{S}$ are exactly the conditional assertions of System P for a particular finite ϵ, and any set of conditional assertions $\Delta = \bigcup_i \{\alpha_i \mid\!\sim \beta_i\}$ will have a corresponding set of *dwff*s $\Delta' = \bigcup_i \{\alpha_i \Rightarrow_{\epsilon_i} \beta_i\}$. We say that Δ' is the *default dual* of Δ, and Δ is the *assertion dual* of Δ'. We apply the same terminology to single *dwff*s and conditional assertions. The reason for writing the defaults in this way is to distinguish between the conditional assertions themselves, and the consequence relation which defines what may be inferred from them—a distinction which is not always clear in work on System P. Assuming that we have a knowledge base Δ which consists of a set of *dwff*s, we can then define the valid set of conclusions which may be drawn from Δ as those sanctioned by the consequence relation $\mid\!\sim_{\mathrm{P}}$ defined in Figure 2[1].

The proof rules that define $\mid\!\sim_{\mathrm{P}}$ may need a little explanation. The rule Ax is a form of "bootstrap" rule which says that if some default $\alpha \Rightarrow_\epsilon \beta$ is in Δ, then were α added to Δ, it would be possible to infer β with probability not less than $1-\epsilon$. The rule And says that if adding α to Δ makes it possible to infer β with probability no less than $1-\epsilon_1$ and γ with probability no less than $1-\epsilon_2$, then adding α to Δ makes it possible to infer $\beta \wedge \gamma$ with probability no less than $1-(\epsilon_1+\epsilon_2)$. Thus the denotation of the consequence:

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon)$$

[1] Note that this includes the two rules Cut and S which can be derived from the basic set of rules.

$$\frac{\alpha \Rightarrow_\epsilon \beta \in \Delta}{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon)} \quad \text{Ax} \qquad \Delta, \alpha \mid\!\sim_{\mathrm{P}} (\alpha, 0) \quad \text{Ref}$$

$$\frac{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon_1) \qquad \Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_2)}{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta \wedge \gamma, \epsilon_1 + \epsilon_2)} \quad \text{And} \qquad \frac{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon_1) \qquad \Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_2)}{\Delta, \alpha \wedge \beta \mid\!\sim_{\mathrm{P}} (\gamma, \frac{\epsilon_2}{1-\epsilon_1})} \quad \text{CM}$$

$$\frac{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon_1) \qquad \beta \vdash \gamma}{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1)} \quad \text{RW} \qquad \frac{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1) \qquad \vdash \alpha \leftrightarrow \beta}{\Delta, \beta \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1)} \quad \text{LLE}$$

$$\frac{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1) \qquad \Delta, \beta \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_2)}{\Delta, \alpha \vee \beta \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1 + \epsilon_2)} \quad \text{Or} \qquad \frac{\Delta, \alpha \wedge \beta \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1)}{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta \rightarrow \gamma, \epsilon_1)} \quad \text{S}$$

$$\frac{\Delta, \alpha \wedge \beta \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1) \qquad \Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon_2)}{\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma, \epsilon_1 + \epsilon_2)} \quad \text{Cut}$$

Figure 2: The consequence relation $\mid\!\sim_{\mathrm{P}}$.

is that on the basis of what is given in Δ, we can infer $\Pr(\beta \mid \alpha) \geq 1 - \epsilon$.

The rules RW and LLE are a little unusual in that both have antecedents which involve $\vdash$, which stands for the consequence relation of standard propositional calculus. Thus RW says that you can replace any inference made by $\mid\!\sim_{\mathrm{P}}$ with any logical consequence, and LLE says that you can replace anything on the lefthand side of $\mid\!\sim_{\mathrm{P}}$ with something that is logically equivalent to it.

This proof system we will call $\mathcal{SP}$. As with any proof system we are interested in the soundness and completeness of the conclusions which may be drawn using $\mathcal{SP}$. We define:

Definition 4 *A* default base *is a set of default well-formed formulae.*

Definition 5 *A basic well-formed formula β is a* p-consequence *of a default base Δ,* conditional on α*, iff:*

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\beta, \epsilon)$$

The value ϵ is known as the *strength* of the consequence. With these definitions, suitable soundness results are easy to obtain. The first relates what can be inferred using $\mid\!\sim_{\mathrm{P}}$ to System P:

Theorem 6 *For every p-consequence β, conditional on α of a default base Δ, $\alpha \mid\!\sim \beta$ is p-entailed by the set of assertions Δ' which is the assertion dual of Δ.*

Proof: $\mathcal{SP}$ has a set of proof rules which mirror those of System P, and anything that may be proved using these rules is a p-consequence. Since in (Kraus, Lehmann, & Magidor 1990) it is shown that anything proved using the rules of System P from a given set of conditional assertions Δ' is p-entailed by that set, it follows that any p-consequence of Δ, the default dual of Δ', is p-entailed by Δ'.

Thus $\mathcal{SP}$ allows us to infer exactly the same things as System P. We also need to show the soundness of the mechanism for propagating the strength of the consequences. This is given by the following:

Theorem 7 *The strengths of the p-consequences of a default base are those justified by probability theory.*

Proof: The soundness of the propagation of ϵ values with respect to probability theory is proved in (Bourne & Parsons 1998).

Together these two results guarantee that $\mathcal{SP}$ is sound—it will generate conclusions sanctioned by System P with probabilistically correct strengths. Since Kraus *et al.* show that the rules of System P are sufficient to infer all the consequences of System P, the following completeness result is immediate:

Theorem 8 *For every $\alpha \mid\!\sim \beta$ which is p-entailed by a set of conditional assertions Δ, β is a p-consequence of the default dual of Δ conditional on α.*

What this theorem guarantees is any conditional assertion which is p-entailed by a given set of defaults will, when those defaults are translated into the language of $\mathcal{SP}$, be a consequence of the corresponding set of *dwff*s. However, this result gives no clue as to the kinds of conclusions we can draw from a given set of *dwff*s. It does not tell us if a particular p-consequence will be found, it just says that it will be found if its assertion dual is p-entailed.

What we would also like are results which say exactly what kind of conclusions we can infer from some initial set of defaults, which therefore give us some idea of what it is reasonable to establish from some initial set of values, and that is what we consider in the remainder of the paper.

Defining the scope of $\mathcal{SP}$

We start by considering that we have a set of *simple defaults* of the form $\alpha \Rightarrow_{\epsilon_i} \gamma_i$ which all have the same antecedent. These form a simple default base:

Definition 9 *A* simple default base *for a language $\mathcal{L}_{\mathcal{S}}$ is a default base:*

$$\Delta = \bigcup_{i=1,\ldots,n} \{\alpha \Rightarrow_{\epsilon_i} \gamma_i\}$$

where α *and the* γ_i *are* bwffs *in* $\mathcal{L}_{\mathcal{S}}$*.*

We can think of the consequents of this set of defaults forming a set $\mathbf{G}$. In general, we have:

Definition 10 *The* consequent set *of a simple default base* Δ *is the set* $\mathbf{G}$ *such that:*

$$\mathbf{G} = \{\gamma_i | \{\alpha \Rightarrow_{\epsilon_i} \gamma_i\} \in \Delta\}$$

and this set $\mathbf{G}$ has what we will call an *associated conjunction* Γ which is a conjunction of all the γ_i in $\mathbf{G}$. Now, applying Ax and CM to $\alpha \Rightarrow_{\epsilon_1} \gamma_1$ and $\alpha \Rightarrow_{\epsilon_2} \gamma_2$, we obtain:

$$\Delta, \alpha \wedge \gamma_1 \mid\!\sim_{\mathrm{P}} \left(\gamma_2, \frac{\epsilon_2}{1-\epsilon_1}\right) \qquad (1)$$

Using the same rules on $\alpha \Rightarrow_{\epsilon_1} \gamma_1$ and $\alpha \Rightarrow_{\epsilon_3} \gamma_3$ gives:

$$\Delta, \alpha \wedge \gamma_1 \mid\!\sim_{\mathrm{P}} \left(\gamma_3, \frac{\epsilon_3}{1-\epsilon_1}\right) \qquad (2)$$

and combining the latter with (1) will give:

$$\Delta, \alpha \wedge \gamma_1 \wedge \gamma_2 \mid\!\sim_{\mathrm{P}} \left(\gamma_3, \frac{\epsilon_3}{1-\epsilon_1-\epsilon_2}\right)$$

If we imagine repeating this process it is clear that given Δ we can recursively apply CM to obtain, for some ϵ:

$$\Delta, \alpha \wedge \mathrm{B}' \mid\!\sim_{\mathrm{P}} (\gamma_i, \epsilon)$$

for any $\gamma_i \in \mathbf{G}$, and for any B' which is the associated conjunction of a set $\mathbf{B}'$ such that $\mathbf{B}' \subseteq \mathbf{G}$. Since And makes it possible to build up conjunctions on the consequent side, similar reasoning makes it obvious that recursively applying the And rule to the same initial set of defaults will give:

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\Gamma', \epsilon')$$

where Γ' is the associated conjunction of a set $\mathbf{G}'$ and $\mathbf{G}' \subseteq \mathbf{G}$. Clearly, then, if we use both rules together, we can derive conclusions of the form:

$$\Delta, \alpha \wedge \mathrm{B}' \mid\!\sim_{\mathrm{P}} (\Gamma', \epsilon'')$$

where B' and Γ' are the associated conjunctions of sets $\mathbf{B}'$ and $\mathbf{G}'$ such that $\mathbf{B}' \subseteq \mathbf{G}$ and $\mathbf{G}' \subseteq \mathbf{G}$. Note it is possible that $\mathbf{B}' \cap \mathbf{G}' \neq \emptyset$. Thus we have:

Theorem 11 *Given a simple default base with antecedent* α *and consequent set* $\mathbf{G}$*, the consequence relation* $\mid\!\sim_{\mathrm{P}}$ *will generate all consequences:*

$$\Delta, \alpha \wedge \mathrm{B}' \mid\!\sim_{\mathrm{P}} (\Gamma', \epsilon)$$

for some ϵ*, where* B' *and* Γ' *are the associated conjunctions of sets* $\mathbf{B}'$ *and* $\mathbf{G}'$ *such that* $\mathbf{B}' \subseteq \mathbf{G}$ *and* $\mathbf{G}' \subseteq \mathbf{G}$*.*

Proof: This follows directly from the above discussion.

This result characterises the kind of consequences we can prove using CM and And on a set of simple defaults. It is possible to generalise these results to wider sets of defaults. Consider that instead of a set of simple defaults, we have, instead, a general set of *conjunctive defaults* of the form $\alpha \wedge \mathrm{B}_i \Rightarrow_{\epsilon_i} \Gamma_i$ where α, as before, is a single proposition and B_i and Γ_i are conjunctions of propositions. This set of defaults is a conjunctive default base:

Definition 12 *A* conjunctive default base *for a language* $\mathcal{L}_{\mathcal{S}}$ *is a default base:*

$$\Delta = \bigcup_{i=1,\ldots,n} \{\alpha \wedge \mathrm{B}_i \Rightarrow_{\epsilon_i} \Gamma_i\}$$

where α *is a* bwff *in* $\mathcal{L}_{\mathcal{S}}$*, and the* B_i *and* Γ_i *are conjunctions of such* bwffs*.*

The formula α is known as the *base antecedent* of the conjunctive default base, and for each default, B_i is the *conjunctive antecedent* and Γ_i is the *consequent.* We distinguish two subsets of Δ, the *simple* subset, which is the set of all simple default rules in Δ. Applying the rules Ax and S to any conjunctive default in Δ will give:

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\mathrm{B}_i \rightarrow \Gamma_i, \epsilon_i)$$

Now, if we can obtain:

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\mathrm{B}_i, \epsilon_j)$$

Applying And will gives us

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\mathrm{B}_i \wedge (\mathrm{B}_i \rightarrow \Gamma_i), \epsilon_i + \epsilon_j)$$

Now, RW makes it possible to replace any p-consequence with any of its logical consequences. This makes it possible to obtain:

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\mathrm{B}_i \wedge \Gamma_i, \epsilon_i + \epsilon_j)$$

and hence:

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\varphi_{i_j}, \epsilon_i + \epsilon_j)$$

for any $\varphi_{i_j} \in \mathbf{B}_i \cup \mathbf{G}_i$ where B_i and Γ_i are the associated conjunctions of $\mathbf{B}_i$ and $\mathbf{G}_i$. This gives us:

Theorem 13 *Given a conjunctive default base* Δ *with base antecedent* α*, whose simple subset has consequent set* $\mathbf{G}$*, the consequence relation* $\mid\!\sim_{\mathrm{P}}$ *will generate all the consequences of the form:*

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\gamma^*, \epsilon)$$

for some ϵ*, where* Δ *contains a default* $\alpha \wedge \mathrm{B}_i \Rightarrow_{\epsilon_i} \Gamma_i$*,* Γ_i *and* B_i *are the associated conjunctions of* $\mathbf{G}_i$ *and* $\mathbf{B}_i$ *respectively,* $\gamma^* \in \mathbf{B}_i \cup \mathbf{G}_i$ *and* $\mathbf{B}_i \subseteq \mathbf{G}$*.*

Proof: From the previous discussion the theorem follows provided that we can infer $\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\mathrm{B}_i, \epsilon_j)$. By Theorem 11, this is possible if $\mathbf{B}_i \subseteq \mathbf{G}$.

What this tells us is that we can use the simple subset of a set of defaults to effectively[2] break down more complex defaults into simple defaults. We can then use these to build up more complex p-consequences just as in Theorem 11:

[2]We use the term "effectively" since we don't obtain new simple defaults, but p-consequences which we would get from simple defaults.

(i)	$\Delta, linda \mid\!\sim_{\mathrm{P}} (steve, 0.1)$	$Ax, 4$
(ii)	$\Delta, linda \mid\!\sim_{\mathrm{P}} (great, 0.01)$	$Ax, 2$
(iii)	$\Delta, linda \wedge steve \mid\!\sim_{\mathrm{P}} (great, 0.011)$	$CM, (i), (ii)$
(iv)	$\Delta, linda \wedge steve \mid\!\sim_{\mathrm{P}} (\neg noisy, 0.05)$	$Ax, 5$
(v)	$\Delta, linda \wedge steve \mid\!\sim_{\mathrm{P}} (great \wedge \neg noisy, 0.061)$	$And, (iii), (v)$

Figure 3: The proof of a conservative consequence about Linda

Theorem 14 *Given a conjunctive default base* Δ *with base antecedent* α*, whose simple subset has consequent set* $\mathbf{G}$*, the consequence relation* $\mid\!\sim_{\mathrm{P}}$ *will generate all consequences:*

$$\Delta, \alpha \wedge \mathrm{B}^{+} \mid\!\sim_{\mathrm{P}} (\Gamma^{+}, \epsilon)$$

for some ϵ *where* B^{+} *and* Γ^{+} *are the associated conjunctions of* $\mathbf{B}^{+}$ *and* $\mathbf{G}^{+}$ *respectively, and both* $\mathbf{B}^{+} \subseteq \mathbf{G}$ *and* $\mathbf{G}^{+} \subseteq \mathbf{G}$*.*

Proof: Immediate by applying the same reasoning as in Theorem 11 to Theorem 13.

We call these Γ^{+} the *conservative consequences* of Δ. Thus the conservative consequences are all the Γ^{+} which are p-consequences of Δ conditional on α and propositions which are themselves the consequences of simple defaults.

Theorems 11 and 14 complement Theorem 8. The latter says that anything provable will eventually be proved. It therefore defines what is provable from above. The former are a first step towards defining what is provable from below—we can prove any conservative consequence, along with a set of other things which follow from the application of other rules. For example we have:

Theorem 15 *Given a conjunctive default base* Δ *with base antecedent* α *whose simple subset has consequent set* $\mathbf{G}$*, the consequence relation* $\mid\!\sim_{\mathrm{P}}$ *will generate all consequences:*

$$\Delta, \top \mid\!\sim_{\mathrm{P}} (\Phi, \epsilon)$$

for some ϵ*, where* Φ *is the associated conjunction of a set* $\mathbf{F}$ *such that* $\mathbf{F} \subseteq \{\alpha\} \cup \mathbf{G}$ *where* $\mathbf{G}$ *is the consequent set of the simple subset of* Δ*.*

Proof: From Theorem 11, we get $\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\Phi, \epsilon)$ provided that $\mathbf{F} \subseteq \mathbf{G}$. Using LLE on this gives us $\Phi \wedge \alpha$ conditional on $\top$ and then applying RW will give us the result.

We also have:

Theorem 16 *Given a conjunctive default base* Δ *with base antecedent* α*, the consequence relation* $\mid\!\sim_{\mathrm{P}}$ *will generate all consequences:*

$$\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\Phi, \epsilon)$$

for some ϵ*, where* $\Phi = \bigwedge_j \varphi_j$ *and:*

$$\bigwedge_i (\neg \mathrm{B}_i \vee \Gamma_i) \vdash \varphi_j$$

where $\Delta = \bigcup_i \{\alpha \wedge \mathrm{B}_i \Rightarrow_{\epsilon_i} \Gamma_i\}$*.*

Proof: For every $\alpha \wedge \mathrm{B}_i \Rightarrow_{\epsilon_i} \Gamma_i$ we can apply Ax, S and RW to get $\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\neg \mathrm{B}_i \vee \Gamma_i, \epsilon_i)$, and applying And to all of these gives $\Delta, \alpha \mid\!\sim_{\mathrm{P}} (\neg \mathrm{B}_i \vee \Gamma_i, \epsilon)$, and applying RW again gives the result.

Further results are possible when we consider proofs which make use of defaults from more than one conjunctive default base. It is also possible to extend the results here by computing bounds on the value of ϵ in the theorems. Lack of space prevents us from giving these additional results here, but they may be found in (Parsons & Bourne 1999).

An example

We now illustrate the use of the system on the following, inspired by examples given by Kraus *et al.* (Kraus, Lehmann, & Magidor 1990).

> Brian and Linda are two happy-go-lucky people who are normally the life and soul of any party (so if either go to a party it will normally be great). Until recently Brian and Linda were married, but then Linda ran off with a mime artist, Steve. As a result, if both Brian and Linda go to the same party they will probably have a screaming row and ruin it (so it will not be great and it will be noisy).
>
> If Linda goes to a party she will probably take her new boyfriend Steve and get him to entertain the guests with his marvellous miming. Thus if Linda goes to a party, Steve will probably go to the same party and if Linda and Steve go to a party together it will normally not be noisy because everyone will be watching his miming. Normally parties that great are noisy and those that are not noisy are not great.

We represent this by the following default base Δ which is made up of four separate default bases. It should be understood that we are trying to ascertain the likelihood of any given party having various attributes (*brian* is present, it is *noisy*, and so on).

1. $brian \Rightarrow_{0.01} great$
2. $linda \Rightarrow_{0.01} great$
3. $brian \wedge linda \Rightarrow_{0.15} \neg great \wedge noisy$
4. $linda \Rightarrow_{0.1} steve$
5. $linda \wedge steve \Rightarrow_{0.05} \neg noisy$
6. $great \Rightarrow_{0.1} noisy$
7. $\neg noisy \Rightarrow_{0.1} \neg great$

(i)	$\Delta, linda \wedge steve \mid\!\sim_{\mathrm{P}} (\neg noisy, 0.05)$	$Ax, 4$
(ii)	$\Delta, linda \mid\!\sim_{\mathrm{P}} (steve, 0.1)$	$Ax, 5$
(iii)	$\Delta, linda \mid\!\sim_{\mathrm{P}} (\neg noisy, 0.15)$	$Cut, (i), (ii)$
(iv)	$\Delta, linda \mid\!\sim_{\mathrm{P}} (great, 0.01)$	$Ax, 2$
(v)	$\Delta, linda \mid\!\sim_{\mathrm{P}} (great \wedge \neg noisy, 0.16)$	$And, (iii), (iv)$
(vi)	$\Delta, \top \wedge linda \mid\!\sim_{\mathrm{P}} (great \wedge \neg noisy, 0.16)$	$LLE, (v)$
(vii)	$\Delta, \top \mid\!\sim_{\mathrm{P}} (\neg linda \vee (great \wedge \neg noisy), 0.16)$	$S, (vi)$
$(viii)$	$\Delta, \neg noisy \mid\!\sim_{\mathrm{P}} (\neg great, 0.1)$	$Ax, 7$
(ix)	$\Delta, \top \wedge \neg noisy \mid\!\sim_{\mathrm{P}} (\neg great, 0.1)$	$LLE(viii)$
(x)	$\Delta, \top \mid\!\sim_{\mathrm{P}} (\neg great \vee noisy, 0.1)$	$S, (ix)$
(xi)	$\Delta, \top \mid\!\sim_{\mathrm{P}} ((\neg great \vee noisy) \wedge (\neg linda \vee (great \wedge \neg noisy)), 0.26)$	$And, (vii), (x)$
(xii)	$\Delta, \top \mid\!\sim_{\mathrm{P}} (\neg linda, 0.26)$	$RW, (xi)$

Figure 4: A non-conservative consequence concerning Linda

As an example of the generation of a conservative consequence, consider the proof of Figure 3. As this proof demonstrates, we can conclude that if both Linda and Steve go to the party, then the probability that it will be both great and not noisy is greater than 0.939 (1 minus the strength of the p-consequence *linda* $\wedge$ *steve*).

If we combine defaults from the different conjunctive default bases in Δ, we can obtain additional conclusions. For example, consider Figure 4 which gives a proof for the p-consequence *linda* conditional on $\top$. This tells us that the probability of Linda going to any particular party is at most 0.26. This last example neatly illustrates two points.

The first is a property of System P. We have shown that the probability of Linda going to any particular party is quite low. It certainly isn't likely enough to be a default conclusion. However, if we know that Linda *does* go to a party—a fact which makes the party somewhat abnormal—then we can draw conclusions which are very likely for such abnormal parties (they are very likely to be great, for instance).

The second point is to do with the form of the proof. As stated above, the proof of the p-consequence $\neg$*linda* involves the use of defaults from different conjunctive default bases (in particular that with base antecedent *linda* and the single default with base antecedent $\neg$*noisy*). This is possible through the use of LLE and S to obtain p-consequences conditional on $\top$ which may then be combined using And. This turns out to be an important mechanism for combining defaults from different default bases (Parsons & Bourne 1999).

Conclusion

This paper has presented some results concerned with the use of System P for generating proofs of propositions. In particular we have defined a system $\mathcal{SP}$ for generating the consequences of a set of defaults expressed using System P. We have shown that this system is sound and complete, and have made a start at precisely characterising the kinds of consequences that this system will generate. The result is that we have identified a small but useful class of consequences of a set of defaults which we know we can prove. Larger classes of such consequences are given in a longer version of this paper (Parsons & Bourne 1999).

The information input into the proof process is a set of lower bounds on conditional probabilities. Because these values are propagated through the proof, the output is a set of probability statements similar to:

$$\Pr(\alpha \wedge \neg\beta \mid \gamma \wedge \delta)$$

If the propositions γ and δ are pieces of evidence (in other words things which are known to have occurred), this output information is sufficient to establish the probability of the state $\alpha \wedge \neg\beta$. Thus the output of $\mathcal{SP}$ can be used, along with information on the utility of $\alpha \wedge \neg\beta$ as the basis of some decision making process, and this is the direction that our research on the topic of this paper is taking us.

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References

Adams, E. 1975. *The Logic of Conditionals*. Dordrecht, Netherlands: Reidel.

Bourne, R., and Parsons, S. 1998. Propagating probabilities in System P. In *Proceedings of the 11th International FLAIRS Conference*, 440–445.

Gabbay, D. 1996. *Labelled Deductive Systems*. Oxford, UK: Oxford University Press.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* 44:167–207.

Parsons, S., and Bourne, R. 1999. On proofs in System P. Technical report, Department of Electronic Engineering, Queen Mary & Westfield College.

Pearl, J. 1988. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. San Mateo, CA: Morgan Kaufmann.