Qualitative, semiqualitative and interval algebras, and their application to engineering problems

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Abstract

This paper proposes the use of semiqualitative modelling for reasoning about the behaviour of complex physical systems. Semiqualitative modelling is a generalisation of qualitative modelling which refines the set of intervals that values may be expressed in. Semiqualitative algebras are introduced, their most important features discussed, and related to qualitative algebras. The advantages that semiqualitative modelling offers over qualitative modelling are demonstrated by the solution of an example from the field of biotechnology. Finally interval algebras are introduced as a generalisation of semiqualitative algebras, and it is proved that it is possible to switch between different interval algebras in the course of computation in order to preserve the greatest possible degree of precision.

Keywords: Qualitative modelling, semiqualitative modelling, interval arithmetic, physical systems, biotechnology.

1. Introduction

Qualitative reasoning was introduced[2] as a formalism for reasoning about physical systems that captures many of the features of human reasoning. Although it is a relatively new idea, it has rapidly become a well established means of analysing the behaviour of physical systems. It is especially useful for modelling complex processes where the governing equations are known in sparse detail since the exact value of physical constants are not required. In general is is only necessary to determine whether values are greater than, equal to, or less than some other value. Indeed, if more detailed information is available, it is not used by the qualitative model; instead precisely known quantities are degraded into qualitative ones. This degrading can cause the qualitative model to be so abstract that either no solutions or many vacuous ones are generated.

This tendency towards overabstraction has prompted research into ways of making qualitative reasoning more precise. In particular Raiman[10] and Mavrovouniotis and Stephanopolous[7] have addressed the loss of information about quantitative magnitude proposing formalisms that explicitly make use of the relative magnitude of values. Such an approach has become known as order of magnitude reasoning, and attempts to model human reasoning of the form:

"if A is much bigger than B, and B is roughly the same size as C, then A is much bigger than C."

Another way of making use of any available quantitative information is adopted in semiqualitative reasoning. Semiqualitative reasoning is a generalisation of qualitative reasoning in which the set of real numbers is broken up into a finite number of intervals. In this paper the idea of semiqualitative reasoning based upon semiqualitative algebras is introduced. Just as Williams[13] defines his qualitative algebra Q1 as a set of values and a set of arithmetic operations over those values, we define a family of semiqualitative algebras as a family of sets of intervals and operations over those intervals. The algebras may then be used for semiqualitative modelling in much the same way as qualitative algebras are used for qualitative modelling.

The paper begins with a brief description of qualitative reasoning which can serve both as an introduction to the concepts for those unfamiliar with the subject, as well as providing a comparison with the semiqualitative methods. This is followed with a simple example of qualitative modelling. Next semiqualitative algebras are introduced, with stress on the links with other formalisms, and the same example is solved in greater detail as befits the more precise formalism. Lastly generalisations of semiqualitative algebras, known as interval algebras, are considered, and ways in which they might be useful are discussed.

2. Qualitative reasoning

Qualitative reasoning reduces the quantitative precision of behavioural descriptions whilst retaining crucial distinctions. Real valued variables are replaced with qualitative variables which can adopt only a small number of values, usually +, 0 and -. The behaviour of a physical system is described in terms of changes in the qualitative value of a number of state variables and their first and second derivatives. These values are related by means of qualitative differential equations, often called confluences. In theory there is no reason to limit the information used to just the first two derivatives, but in practice it is extremely difficult to obtain higher order derivatives. All time derivatives are continuous, so that no variable may jump from one qualitative state to another without passing through any intervening states, and variables are combined by means of combinator tables giving the result of every possible combination of inputs. For a more detailed discussion of qualitative methods see Davis[4] and Weld and de Kleer[12].

2.1 Qualitative algebras

A qualitative algebra is a set of qualitative operands, the full set of values that a qualitative variable and its derivatives may take on, and the set of operations over those values. The simplest possible qualitative algebra has operands {+, 0, -}, and the operations $\otimes$ and $\oplus$, qualitative multiplication and addition respectively. We may formally define such an algebra (after Williams[13]) as operating on the set S = {-, 0, +} whose members denote the sign of real quantities. The relation between $\Re$, the set of real numbers and S is defined by the mapping $[\cdot]: \Re \rightarrow S$ where:

$$\text{For any } x \in \Re,\ [x] = \begin{cases} + & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ - & \text{if} \quad x < 0 \end{cases} \qquad [1]$$

Thus the operator $[\cdot]$ partitions $\Re$ into three intervals, [0, +•], [0, 0] and [-•, 0] which correspond to +, 0, -. This set is extended to include the value ? which represents an indeterminate sign corresponding to the interval [-•, 0, •], forming a new set S' = {-, 0, +, ?}. The operations over the algebra are {$\otimes$, $\oplus$}. The operator $\oplus$: S' × S' → S' is the qualitative analog of addition on reals, and returns the sign of x + y, deduced from the sign of x and y. This is summarised by the combinator table in Figure 1. ⊗ is similarly defined.

⊕	+	0	-
+	+	+	?
0	+	0	-
-	?	-	-

⊗	+	0	-
+	+	0	-
0	0	0	0
-	-	0	+

Figure 1.

2.2 Qualitative modelling

This section presents a two tier introduction to the idea of qualitative modelling. Firstly a brief description for those readers uninterested in the detail of the subject is given, introducing the method by means of a simple example. Then there is a more detailed discussion which leads on to the solution of a more complex example in Section 2.3.

2.2.1 A first description

A basic grasp of qualitative modelling may be obtained from an intuitive understanding of the concepts described in this section. Rather than presenting a discussion of the various issues at stake, we analyse a simple example. The only values that we consider as quantifiers for the values of a quantity, and the values of its first and second derivatives are the + (positive), 0 (zero) and - (negative) of [1]. Since third and higher derivatives are usually unavailable, any qualitative variable is fully specified by the triplet <X, DX, DDX> of value, first derivative and second derivative.

As an example of a qualitative model, consider the following set of equations:

$$x_1 \oplus x_2 = x_3 \qquad [2]$$

$$x_1 \otimes x_4 = x_3 \qquad [3]$$

$$\frac{dx_4}{dt} = x_5 \qquad [4]$$

The total number of qualitative variables is 5. The following set of 5 triplets is one possible assignment of value, first and second derivative to the five variables:

x_1	x_2	x_3	x_4	x_5
<+ + +>	<+ + -> <- - ->	<+ + +>	<+ + 0>	

The state of the system described by this assignment is not a solution of the set of equations [2–4] since x_3 is determined from x_1 and x_2 , both of which are +, by equation [2], and + ⊕ + = +, whereas x_3 = -. By similar means it is possible to identify all the 5-triplets which are solutions of the set of equations, and these correspond to all the qualitative solutions of the model. This brute force approach provides correct solutions, but its effectiveness is limited by the complexity of the models. In particular enumerating every possible second derivative of a product of several variables is extremely time consuming.

2.2.2 A more detailed description

A qualitative model of a system is a set of equations in which a set of variables and constants whose values are drawn from the set {+, 0, -} and whose behaviour defines the the behaviour of the system, are related using the operations of the qualitative algebra. A solution of the set of equations represents a particular way in which the system may behave. A full description includes the dynamic as well as the static behaviour of the variables, and so the derivatives of variables are commonly included in the equations. Any variable X in a qualitative model is fully specified if the triplet <X, DX, DDX> of value, first derivative and second derivative is given for all inputs and outputs. The triplet may be made more specific by using the third and higher derivatives. However in realistic situations, the third derivative is unlikely to be available since there is simply insufficient information about the kind of complex physical systems for the analysis of which qualitative reasoning is appropriate to enable third and higher derivatives to be assessed.

Any qualitative system may be described in terms of the qualitative states which it may reside in. A state is an assignment of a single qualitative value to each of the variables that define the behaviour of the system and their first and second derivatives. Thus if a set of n different qualitative variables describe a particular system:

$$X_{1)}, X_{(2)}, \ldots, X_{(n)}$$

then the specification of a qualitative state is a set of n triplets:

$$<X_{(1)},DX_{(1)},DDX_{(1)}>,<(X_{(2)},DX_{(2)},DDX_{(2)}>,\ldots,<X_{(n)},DX_{(n)},DDX_{(n)}>$$

Any qualitative state is a possible solution of a qualitative model. Since each member of each triplet can take on one of three values, there are 27^n different qualitative states. The set of equations that comprise the qualitative model rule out certain of these solutions, and can therefore be considered as constraints against which possible solutions are tested.

Since the number of possible qualitative states is so large, the number of solutions of a given qualitative model is often extremely large. To further reduce this number it is possible to specify additional constraints; specifying the exact value, derivative and second derivative of particular variables. It is also possible to ignore particular variables which, although they must be included in the model, are not of primary interest.

2.3 Modelling a bioreactor

As an example consider the qualitative modelling of a bioreactor[1] used for the treatment of biologically degradable waste. A slightly simplified model of the bioreactor is detailed by the following set of differential equations:

$$\frac{dx_1}{dt} + (k_{12} + k_{13}).x_1 + k_{11}x_1.\, x_5 = + k_{21}.x_2$$

$$\frac{dx_2}{dt} + k_{21}\, x_2 = k_{12}\, x_1$$

$$\frac{dx_3}{dt} = k_{13}\, x_1 + k_{43}\, x_4$$

$$\frac{dx_4}{dt} + k_{43}\, x_4 = k_{11}\, x_1\, x_5$$

$$\frac{dx_5}{dt} + k_{53}x_5 + k_{11}\, x_1\, x_5 = 0$$

where x_1–x_5 are concentrations of various substrates, either those wastes being digested or the products of the digestion. Writing dx as a shorthand for dx/dt, and ddx as shorthand for $\frac{d^2x}{dt^2}$, we can specify additional constraints as:

$$x_1 = +,\ dx_1 = -,\ ddx_1 = 0,\ dx_5 = +,\ ddx_5 = 0$$

This set of constraints may be considered as a query, in this case asking the question *"When x_1 is present (x_1 = +), what are the ways in which it is possible to achieve a linear (ddx_1 = 0) decrease (dx_1 = -) of concentration of x_1 while x_5 is present (x_5 = +) and changes linearly (ddx_5 = 0)?"*. Solving this gives the solution:

x_1	x_2	x_3	x_4	x_5
<+ - 0>	<+ + ->	<+ + +>	<+ + ?>	<+ ? 0>

where x_1 and x_5 are as set by the query. The solution may be interpreted to mean that the value of x_2 is positive and increasing at a diminishing rate, whilst x_3 is positive and increasing at an increasing rate, and x_4 is positive and increasing at a rate that may be increasing (ddx = +), decreasing (ddx = -), or unchanging (ddx = 0). The value of x_2 thus behaves as in Figure 2, and that of x_3 as in Figure 3, and x_4 as in Figure 4.

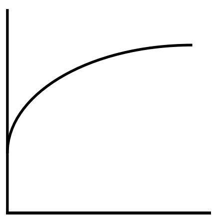


Figure 2.

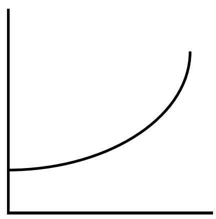


Figure 3.

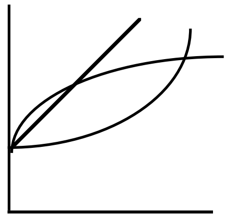


Figure 4.

3. Semiqualitative reasoning

Semiqualitative reasoning is a generalisation of qualitative reasoning which increases precision by splitting the real numbers into a set (2k +1) intervals comprised of a zero interval [0, 0] and 2k continuous intervals that are symmetrical about the zero interval. Clearly the standard quantity space {+, 0, -} is that obtained for k = 1, and it is possible to solve qualitative problems using semiqualitative methods. The size of the set of intervals that a semiqualitative variable may adopt defines the precision with which information may be determined. This precision is related to the degree of uncertainty present in the knowledge.

There are several different methods of choosing the values that define the set of intervals. From experience three of these are of general use, namely an arithmetic sequence of values, a geometric sequence of values, or a set of values chosen by someone familiar with the system being modelled. The third method is the most flexible since the intervals are chosen to reflect the values that are felt to be important. It enables us to specify intervals that are not equidistant, and follow no sequence. The intervals may cluster at extreme ends of the scale, or may group around critical regions. The only restriction is that the intervals must be symmetrical about zero. The chosen set of boundaries can reflect a certain point of view and/or the accuracy of the knowledge that is available.

However, the advantage of the arithmetic sequence is that through providing a vast number of intervals it ensures that adequate coverage is given to all areas of the interval range. This guarantees that the system will operate correctly, albeit a little slowly. There is no such safeguard when the user specified intervals are used, and it is possible that the first choice set of values is not adequate. In this case a trial and error approach is the only alternative with successive choices homing in on a reasonable set of intervals.

For practical applications it is useful to introduce a *coefficient of interval contraction* c. Consider we are adding two values I_i and I_j, where the upper bound of the result of this addition falls is the value I_{s+1}. Now we define c as:

$$\frac{(I_i + I_j)-I_s}{(I_{s+1}-I_s)} \quad < \quad c$$

where:

$$I_s \quad < \quad I_i + I_j \quad < \quad I_{s+1}$$

and we say that provided the inequality holds then the upper boundary for $I_i + I_j$ is not I_{s+1} but I_s. The purpose of this is to introduce a little flexibility into the interval system. If c is zero, then as soon as a value exceeds the upper limit of an interval, by however small an amount, the value is assessed as if it is the higher interval. Clearly this will not always desirable, and setting $c > 0$ defines the proportion of the interval by which a value is allowed to exceed the interval before being classified as belonging to the next interval.Usually the coefficient c is chosen to be of the order of 0.05.

Semiqualitative techniques have been successfully applied to number of engineering problems which are not soluble by pure qualitative reasoning. Such problems include simulations of chemical reactions[6] and bioengineering processes[5].

3.1 Semiqualitative algebras

A semiqualitative algebra is a set of semiqualitative intervals coupled with arithmetic operations over those intervals. A general semiqualitative algebra may be defined as follows. We have a set $\mathbf{I} = \{[I_{-k}, I_{-(k-1)}], [I_{-(k-1)}, I_{-(k-2)}],..., [I_{-1}, I_0], [I_0, I_0], [I_0, I_1],..., [I_{(k-2)}, I_{(k-1)}], [I_{(k-1)}, I_{(k)}]\}$ of 2k+1 intervals, where $I_{-m} = I_m$, and $I_{-k} < I_{-(k-1)} < I_{-(k-2)} <,..., < I_{-1} < I_0 < I_1 <,..., < I_{(k-2)} < I_{(k-1)} < I_{(k)}$, which form the basic operands of the algebra.

We also have a set of arithmetic operations over the set of intervals, $\{\oplus, \ominus, \oslash, \otimes\}$, which are analagous to the operations on real numbers. The properties of these operators have been defined by Moore[8]:

$$[a, b] \otimes [c, d] = [ac, bd] \quad [5]$$
$$[a, b] \oplus [c, d] = [a + c, b + d] \quad [6]$$
$$[a, b] \ominus [c, d] = [a - d, b - c] \quad [7]$$

for a < b, c < d, and:

$$[a, b] \oslash [c, d] = [\frac{a}{d}, \frac{b}{c}] \quad [8]$$

for a < b, c < d, and d, b ≠ 0. Thus when combining two interval values the result is the widest possible interval that the combination of their boundaries can produce. This leads to problems with closure. When, for example, we add [0, 40] to [40, 50] in the interval space {[0, 40], [40, 50], [50, 90]} we find that the solution is the new interval [40, 90].

Thus our operand set must be expanded to $\mathbf{I}^* = \{[I_{-k}, I_{-(k-1)}], [I_{-k}, I_{-(k-2)}],..., [I_{-k}, I_{(k)}], [I_{-(k-1)}, I_{-(k-2)}],..., [I_{-(k-1)}, I_k],..., [I_{(k-1)}, I_{(k)}]\}$ the set of all inetrvals that may be composed from the boundaries of the intervals of set **I**. if we further set $I_{-k} = -\infty$, and $I_k = \infty$, then we have closure for the operations $\{\oplus, \ominus, \oslash, \otimes\}$. The relationship between the set of real numbers $\Re$ and the set **I** is established by the mapping $[[\cdot]]: \Re \rightarrow \mathbf{I}$ where for any $x \in \Re$, $[[x]] = [I_i, I_j]$ such that $I_i < x < I_j$.

3.2 Semiqualitative modelling

Semiqualitative modelling is essentially qualitative modelling with more values. A semiqualitative model of a system is a set of equations in which a set of variables and constants whose values are drawn from the set of semiqualitative intervals are related using the operations of the semiqualitative algebra. As for qualitative modelling, a solution of the set of equations represents a particular way in which the system may behave. The set of intervals are the only quantifiers in semiqualitative reasoning systems, and so not only the values of variables but also the value of their first and second derivatives must be expressed using these intervals. As is the case for qualitative systems, then, semiqualitative modelling consists of establishing possible values of the specified variables and their derivatives from the set of interval operands of the algebra, using the set of equations as constraints.

Since the number of possible semiqualitative states is even larger than the set of qualitative solutions for a particular system, the use of additional constraints specifying the exact value, derivative and second derivative of particular variables is even more important than in qualitative modelling. This problem is further aggravated by the fact that, in comparison with qualitative arithmetic whose operations are simple and quick to carry out, the operations of semiqualitative arithmetic may be quite time consuming.

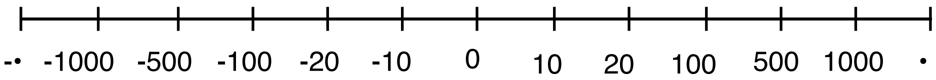
3.3 The bioreactor revisited

Using a semiqualitative model, it is possible to investigate the bioreactor of Section 2.3 in more detail. For instance, it is known that:

$$x_1 = 0\text{–}20$$
$$x_5 = 0\text{–}10$$
$$k_{11} = 100$$
$$\frac{k_{12}}{k_{21}} = 0.5$$

$$k_{13} = 5.0$$
$$k_{43} = 1.0$$
$$k_{53} = 0.3$$

as well as the fact that $k_{12} = 1.5$, and $k_{21} = 3.0$. This model is solved with two sets of semiqualitative intervals. Firstly, the boundaries are set at:



and the coefficient of interval contraction c is set to 5%. When the additional constraints are set as:

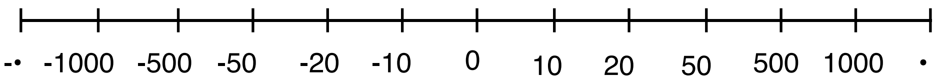
		x	dx	ddx	
x_1	<	(0–20)	(0–10)	0	>
x_2	<	(10–1000)	?	?	>
x_3	<	(0–20)	?	?	>
x_4	<	(10–1000)	?	?	>
x_5	<	(10–20)	(0—10)	0	>

with each triplet being value, first derivative and second derivative as in the qualitative case, solutions such as that below may be obtained. The complexity of the full solution means that it is impractical to list every solution.

		x	dx	ddx	
x_2	<	(20–100)	(0–10)	(-20 —100)	>
x_3	<	(10–20)	(20–100)	(20–100)	>
x_4	<	(0–10)	(0–10)	0	>

From this example it is clear that the solutions of the semiqualitative model are considerably more detailed than those obtained by the qualitative method of Section 2.3, while agreeing with them The results of the semiqualitative analysis enable us to draw up a semiqualitative phase portrait[3]. As the number of semiqualitative intervals increases, so this phase portrait will approach that obtained by an analytical solution to the set of equations, becoming identical as the number of intervals approaches infinity.

Next, the boundaries are set to:



so that the third interval, in which many of the solution values lie is narrowed. With the same set of constraints as before the following solution is generated:

		x	dx	ddx	
x_2	<	(20–50)	(0–10)	(-20 —50)	>
x_3	<	(10–20)	(20–50)	(20–50)	>
x_4	<	(0–10)	(0–10)	0	>

showing how the refinement of the set of semiqualitative intervals may improve the solution.

4. Interval algebras

It is possible to generalise the concept of semiqualitative algebras. The operations of interval arithmetic may be used over any set of intervals to form an *interval algebra*. Thus an interval algebra can be based upon intervals that are not symmetrical about zero, intervals which overlap, or even intervals that are not continuous. An interval algebra is defined over a ordered set of values $V = \{v_1,...,v_n\}$, where $v_1 < v_2 <,...,< v_n$. These are used to define a set of intervals $\mathbf{I}_v = \{[v_i, v_j] : v_i \leq v_j$ and $v_i, v_j \in V\}$, the set of all intervals that may be composed from the set of values V.

Arithemtic operations over this set of intervals are described by interval arithmetic, [5]–[8], and under such operations the set V is closed provided that v_1 = -•, and v_n = •. Interval algebras may have degenerate intervals $[v_i, v_j]$ such that $v_i = v_j$ as operands. When combining two degenerate intervals, interval arithmetic reduces to ordinary arithmetic.

There is a mapping between the real numbers and $\mathbf{I}_v$, $\{[\cdot]\}$: $\Re \rightarrow \mathbf{I}_v$, such that for any $\{[x]\} = [v_i, v_j]$ where $v_i \leq x \leq v_j$ and for all k such that $1 \leq k \leq n$, if $v_k \leq x$, then $v_k \leq v_i$ and if $v_k \geq x$, $v_k \geq v_j$. $\{[\cdot]\}$ thus maps a given number to the smallest interval defined by V that contains it. If closure under arithmetic operations is not a requirement, the intervals need not be continuous, and we can choose an operand set $\mathbf{I}^*_v$ ‹ $\mathbf{I}_v$. In such a case the mapping between $\Re$ and $\mathbf{I}^*_v$ will not be defined for every member of $\Re$. It is possible to define an order $\leq_{Q3}$[9] over the intervals such that $[v_i, v_j] < [v_k, v_l]$ iff $(v_i + v_j) \leq (v_k - v_l)$.

Travé-Massuyès and Piera[11] present a mathematical framework to support reasoning with interval algebras that explicitly distinguishes between different levels of description. Given a set of values S and an order ≤ defined over S, qualitative equality ± is defined as:

a ± b if there exists x ∈ S such that x ≤ a and x ≤ b.

A qualitative algebra is a pair (S, ±) provided with operations ⊕ and ⊗, which are:

(i) qualitatively associative: a ⊗ (b ⊗ c) ± (a ⊗ b) ⊗ c and a ⊕ (b ⊕ c) ± (a ⊕ b) ⊕ c
(ii) qualitatively commutative: a ⊗ b ± b ⊗ a and a ⊕ b ± b ⊕ a [6]
(iii) ⊗ is qualitatively distributive with respect to ⊕: a ⊗ (b ⊕ c) ± (a ⊗ b) ⊕ (a ⊗ c)

Travé-Massuyès and Piera prove that a qualitative algebra (S, ±, ⊕, ⊗) and a subalgebra (T, ±, ⊕, ⊗) where T ‹ S and T ≠ Ø are embedded in one another, and that it is possible to dynamically refine a model during processing by switching from T to S.

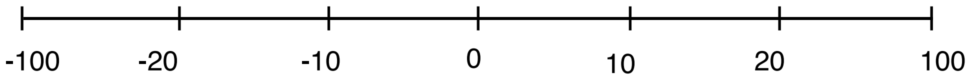
In other words, when combining two interval values whose result is very imprecise due to the width of the interval that their solution lies in, it is possible to dynamically refine the set of intervals to make the answer more precise. The advantage of refining the intervals dynamically rather than starting with a large number of small intervals to begin with is that the dynamic approach reduces the number of intervals which must be considered to a minimum. This in turn reduces the number of possible solutions limiting the time and expense of the computation. Thus it is desirable to show that interval algebras, and thus all semiqualitative algebras, are a family of algebras which may be dynamically refined. We have:

Theorem 4.1: Interval algebras are qualitative algebras (S_i, ±, ⊕, ⊗) for interval addition + = ⊕, interval multiplication × = ⊗, and Q-equality defined by :

> $[v_l - v_j] \pm [v_k - v_i]$ if there exists $[v_a, v_b]$ such that $[v_a, v_b] \leq_{Q3} [v_i, v_j]$ and $[v_a, v_b] \leq_{Q3} [v_k, v_l]$.

Proof: See Appendix.

Consider the basic set of intervals {[-100, 20], [20, 10], [10, 0], [0, 10], [10, 20], [20, 100]}:



It is possible to specify a family S of algebras over the intervals such that <S_i, ±, ⊕, ⊗> ∈ S and S_i ∕ { {[-100, -20], [-20, -10], [-10, 0], [0, 10], [10, 20], [20, 100]}, {[-100, -10], [-20, 0], [-10, 10], [0, 20], [10, 100]}, {[-100, 0], [-20, 10], [-10, 20], [0, 100]}, {[-100, 10], [-20, 20], [-10, 100]} {[-100, 20], [20, 100]}, {[-100, 100]} }. There is an implied order on the S_i that is summarised by Figure 5:

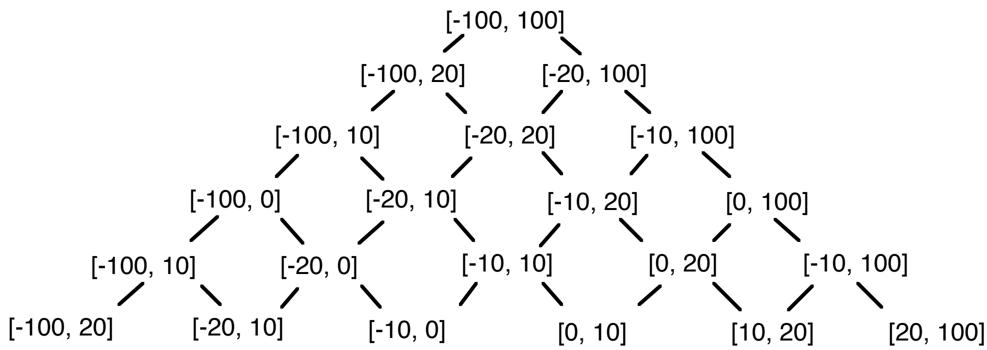


Figure 5.

Since the family S are interval algebras and thus obey Travé-Massuyès and Piera's axioms, it is possible to switch between levels of granularity as required. The advantages of this are as follows. Consider we are using the set of intervals {[-100, 10], [-20, 20], [-10, 100]} and that we have x = y = [-20, 20], and we wnat to determine the value of z = x - y. Clearly the exact value of z by interval arithmetic is [-40, 0]. Using our set of values the smallest interval that we can assign to the value of z is [-100, 10] which is considerably less accurate an interval than the optimum. Now, if we switch to the algebra baased on the set of values {[-100, 0], [-20, 10], [-10, 20], [0, 100]} we can establish z = [-100, 0] which is somewhat better, and indeed represents the best bounds that we can establish on z given the initial set of boundary values. Thus a system employing interval algebras and switching between them when such a move improves the results of arithmetic operations will generate more precise answers than one that does not.

5. Summary

The concepts of semiqualitative and interval algebras have been introduced as successive generalisations of qualitative algebras. Their use in modelling engineering systems has been discussed, and examples of their application to the modelling of a simplified bioreactor have been given. In addition, the theoretical relationships between the algebras have been discussed in some depth, and it has been shown that the dynamic refinement of intervals is possible, allowing reasoning systems constructed using interval and semiqualitative algebras to preserve the tightest desirable bounds on values of interest.

Appendix

We have the following theorem that justifies our using any interval algebra as a qualitative algebra, as defined by Travé-Massuyès and Piera[11], and thus our defining a family of algebras upon it which may be switched between at will:

Theorem 4.1: Interval algebras are qualitative algebras (S_i, ±, ⊕, ⊗) for interval addition + = ⊕, interval multiplication × = ⊗, and Q-equality defined by :

> $[v_l - v_j]$ ± $[v_k - v_i]$ if there exists $[v_a, v_b]$ such that $[v_a, v_b] \leq_{Q3} [v_i, v_j]$ and $[v_a, v_b] \leq_{Q3} [v_k, v_l]$.

Proof: To prove the theorem it is necessary to show that each of the three conditions [6] holds.

(i) To prove the associativity of × we must show ([a, b] × [c, d]) × [e, f] ± [a, b] × ([c, d] × [e, f]). Now, from the definition of interval multiplication [5], we have ([a, b] × [c, d]) × [e, f] = [ace, bdf] = [a, b] × ([c, d] × [e, f]) which means that we have both ([a, b] × [c, d]) × [e, f] $\leq_{Q3}$ [a, b] × ([c, d] × [e, f]) and [a, b] × ([c, d] × [e, f]) $\leq_{Q3}$ ([a, b] × [c, d]) × [e, f] . This in turn means that we have ([a, b] × [c, d]) × [e, f] ± [a, b] × ([c, d] × [e, f]), and the associativity of × is proved. To prove the associativity of + we need to show ([a, b] + [c, d]) + [e, f] ± [a, b] + ([c, d] + [e, f]). From the definition of interval addition [6], we have ([a, b] + [c, d]) + [e, f] = [a + c + e, b + d + f] = [a, b] + ([c, d] + [e, f]) . As above this means that + is associative, and the condition holds.

(ii) To prove the commutativity of × and + we have to show [a, b] × [c, d] ± [c, d] × [a, b] and [a, b] + [c, d] ± [c, d] + [a, b] must hold. [a, b] × [c, d] = [c, d] × [a, b] and [a, b] + [c, d] = [c, d] + [a, b] both follow directly from the definition of interval multiplication [5] and addition [6], and as above this means that the condition holds.

(iii) For this condition to hold we need interval multiplication to distribute over interval addition. Thus we require [a, b] × ([c, d] + [e, f]) ± [(a, b] × [c, d]) + ([a, b] × [e, f])]. Again [a, b] × ([c, d] + [e, f]) = [a(c + e), b(d + f)] = [ac + ae, bd + bf] = [(a, b] × [c, d]) + ([a, b] × [e, f])] follows from the definitions of the arithmetic operations, and this in turn means that multiplication distributes over addition, and all the conditions, and thus the theorem, hold. ■

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