A proof theoretic approach to qualitative probabilistic reasoning

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Abstract

In recent years, several papers have described systems for plausible reasoning which do not use numerical measures of uncertainty. Some of these have been based on logic and some have been based on causal influences. This paper suggests one way of combining the advantages of both types of approach by introducing a means of reasoning with causal influences in a proof theoretic way.

1 Introduction

In the last few years there have been a number of attempts to build systems for reasoning under uncertainty that are of a qualitative nature—that is they use qualitative rather than numerical values, dealing with concepts such as increases in belief and the relative magnitude of values. Between them, these systems address the problem of reasoning in situations in which knowledge is uncertain, but in which there is a limited amount of numerical information quantifying the degree of uncertainty. Three main classes of system can be distinguished—systems of abstraction, infinitesimal systems, and systems of argumentation. In systems of abstraction, the focus is mainly on modelling how the probability of hypotheses changes when evidence is obtained and there is no need to commit to exact probability values. They thus provide an abstract version of probability theory, known as qualitative probabilistic networks (QPNs), which ignores the actual values of individual probabilities but which is nevertheless sufficient for planning [30], explanation [6] and prediction [22] tasks. Similar systems have

also been used to provide an account of default reasoning [17, 18]. Infinitesimal systems deal with beliefs that are very nearly 1 or 0, providing formalisms that handle order of magnitude probabilities. Infinitesimal systems may be used for diagnosis [4] as well as providing a general model of default reasoning [13], and have been extended with infinitesimal utilities to give complete decision theories [26, 32]. Systems of argumentation are based on the idea of constructing logical arguments for and against formulae, establishing the overall validity of such formulae by assessing the persuasiveness of the individual arguments. Systems of argumentation have been applied to a problems such as diagnosis, protocol management and risk assessment [10], as well as handling inconsistent information [2], and providing a framework for default reasoning [9, 16, 27].

This paper provides a hybridisation of the logical and abstraction approaches by introducing a logical approach to reasoning about how probabilities change, which will be called the qualitative probabilistic reasoner ($\mathcal{QPR}$). As is argued below, $\mathcal{QPR}$ provides a more flexible, expressive, and natural means of reasoning about such changes than is currently possible in systems of abstraction. The development of $\mathcal{QPR}$ relies upon a number of results established in the study of QPNs. Such results are not explained in any detail since they are easily found in the literature (in [7] and [30]) and would further lengthen this already lengthy paper.

2 The logical language

2.1 Basic concepts

We start with a set of atomic propositions $\mathcal{L}$. We also have a set of connectives $\{\neg, \wedge, \rightarrow, \uplus, \rightsquigarrow\}$, and the following set of rules for building the well-formed formulae (*wff*s) of the language.

1. If $l \in \mathcal{L}$ then l is a simple well-formed formula (*swff*).
2. If l is an *swff*, then $\neg l$ is an *swff*.
3. If l and m are *swff*s then $l \wedge m$ is an *swff*.
4. If l and m are *swff*s then $l \rightarrow m$ is an implicational well-formed formula (*iwff*).
5. If l, m and n are *swff*s then $l \uplus m \rightsquigarrow n$ is a synergistic well-formed formula (*ywff*).
6. The set of all *wff*s is the union of the set of *swff*s, the set of *iwff*s, and the set of *ywff*s.

There are a couple of points that should be noted about the connectives which go to make up these formulae. The first point is that neither $\rightarrow$ or $\rightsquigarrow$ represents material implication. Instead both represent a constraint on the conditional probabilities relating the formulae they connect. The second point is

that this is not the complete set of connectives which can be handled within the framework—it is also possible to deal with disjunction and material implication [21]—the set is made up of the connectives necessary to capture qualitative probabilistic reasoning of a slightly richer form than that exhibited by Wellman's qualitative probabilistic networks (QPNs) [30].

The set of all *wff*s that may be defined using $\mathcal{L}$, may then be used to build up a database Δ where every item $d \in \Delta$ is a triple $(i : l : s)$ in which i is a token uniquely identifying the database item (for convenience we will use the letter 'i' as an anonymous identifier), l is a *wff*, and s gives information about the probability of l. In particular we take triples $(i : l : \uparrow)$ to denote the fact that $\Pr(l)$ increases, and similar triples $(i : l : \downarrow)$, to denote the fact that $\Pr(l)$ decreases. Triples $(i : l : \leftrightarrow)$, denote the fact that $\Pr(l)$ is known to neither increase nor decrease. It should be noted that the triple $(i : l : \uparrow)$ indicates that $\Pr(l)$ either goes up, or does not change—this inclusive interpretation of the notion of "increase" is taken from QPNs—and of course a similar proviso applies to $(i : l : \downarrow)$. Since we want to reason about changes in belief which equate to the usual logical notion of proof, we also consider changes in belief to 1 and decrease in belief to 0, indicating these by the use of the symbols $\Uparrow$ and $\Downarrow$ and values which are 1 and 0. The meaning of a triple $(i : l : \Uparrow)$ is that the probability of l becomes 1, $(i : l : \Downarrow)$ means that the probability of l becomes 0, $(i : l : 1)$ means that the probability of l is 1 and $(i : l : 0)$ means that the probability of l is 0. We also have triples $(i : l : \updownarrow)$ which indicate that the change in $\Pr(l)$ is unknown. In addition, for reasons which will become clear later, we need a symbol to denote a probability whose value is not known (as distinct from a change in probability whose value is not known). This symbol will be $\wr$, so the triple $(i : l : \wr)$ means that the value of $\Pr(l)$ is unknown. While this profusion of symbols might seem baroque, it is unfortunately necessary in order to distinguish the different aspects of qualitative probabilistic reasoning.

In fact the use of this kind of set of symbols should be familiar from qualitative reasoning [15]. In qualitative reasoning we consider variables and the changes in value of those variables. A given variable x can have a positive or negative value, denoted [+] or [−], and we also distinguish the landmark value [0]. We are also interested in the way x changes over time and handle this by considering the values of dx/dt. These values may also be [+], [0] and [−]. In $\mathcal{QPR}$ the same distinctions are made. We have the probability value $\wr$, which corresponds to [+], and landmark values of 0 and 1. We also have the changes in probability $\uparrow$, $\downarrow$ which correspond to [+] and [−] derivatives, and distinguish the landmark changes $\leftrightarrow$, $\Uparrow$ and $\Downarrow$. The additional symbols obviate the use of explicit derivatives.

2.2 Non-material implication

As mentioned above, $\rightarrow$ does not represent material implication but a connection between the probabilities of antecedent and consequent. This is the key to understanding the system. We take *iwff*s, which we will also call "implications", to denote that the antecedent of the *iwff* has a probabilistic influence on the

consequent. Thus we are not concerned with the probability *of* the *iwff*, but what the *wff* says about the probabilities of its antecedent and consequent. More precisely we take the triple $(i : a \rightarrow c : +)$ to denote the fact that:

$$\Pr(c|a, X) \geq \Pr(c|\neg a, X)$$

for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ (where s is any sign) or $(i : \neg x \rightarrow c : s)$. The effect of the X in this inequality is to ensure that the restriction holds whatever is known about formulae other than c and a—whatever the probabilities of a and c, the constraint on the conditional probabilities holds. Similarly the triple $(i : a \rightarrow c : -)$ denotes the fact that:

$$\Pr(c|a, X) \leq \Pr(c|\neg a, X)$$

again for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ or $(i : \neg x \rightarrow c : s)$. It is possible to think of an implication $(i : a \rightarrow c : +)$ as meaning that there is a constraint on the probability distribution over the formulae c and a such that an increase in the probability of a entails an increase in the probability of c, and an implication $(i : a \rightarrow c : -)$ means that there is a constraint on the probability distribution over the formulae c and a such that an increase in the probability of a entails a decrease in the probability of c. We do not make much use of triples such as $(i : c \rightarrow a : 0)$[1] since they have no useful effect but include them for completeness—$(i : c \rightarrow a : 0)$ indicates that:

$$\Pr(c|a, X) = \Pr(c|\neg a, X)$$

for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ or $(i : \neg x \rightarrow c : s)$, and so denotes the fact that $\Pr(c)$ does not change when $\Pr(a)$ changes. We also have implications such as $(i : a \rightarrow c : ?)$ which denotes the fact that the relationship between $\Pr(c|a, X)$ and $\Pr(c|\neg a, X)$ is not known, so that if the probability of a increases it is not possible to say how the probability of c will change. With this interpretation, implications correspond to qualitative influences in QPNs. Just as in QPNs, we require that implications are causally directed, by which we mean that the antecedent is a cause of the consequent. This is the usual restriction imposed in probabilistic networks [25] and, as will become apparent, is necessary to ensure that the system is sound.

This simple picture is complicated because we have *categorical* implications which allow formulae to be proved true or false. In particular, an implication $(i : a \rightarrow c : ++)$ indicates that when a is known to be true, then so is c. Thus it denotes a constraint on the probability distribution across a and c such that if $\Pr(a)$ becomes 1, then so does $\Pr(c)$. This requires that:

$$\Pr(c|a, X) = 1$$

for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ or $(i : \neg x \rightarrow c : s)$ [19]. Note that this type of implication also conforms to the conditions for

[1] As a result we will not worry about the possibility of confusing $(i : c \rightarrow a : 0)$ with $(i : l : 0)$ where l is an *swff*.

implications labelled with +, and that if $\Pr(c|\neg a, X) = 1$ then $\Pr(c)$ is always equal to $\Pr(a)$. Similarly, a probabilistic interpretation of an implication $(i : a \rightarrow c : --)$ which denotes the fact that if a is true then c is false requires that:

$$\Pr(c|a, X) = 0$$

for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ or $(i : \neg x \rightarrow c : s)$. The conditions imposed on the conditional values by these implications suggest the existence of a further pair of types of categorical implication which are symmetric to those already introduced. We have an implication $(i : a \rightarrow c : -+)$ which denotes the constraint:

$$\Pr(c|\neg a, X) = 1$$

for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ or $(i : \neg x \rightarrow c : s)$, and an implication $(i : a \rightarrow c : +-)$ which denotes the constraint:

$$\Pr(c|\neg a, X) = 0$$

for all $X \in \{x, \neg x\}$ for which there is a triple $(i : x \rightarrow c : s)$ or $(i : \neg x \rightarrow c : s)$.

2.3 Synergy

Being able to handle synergy relations is an important part of any qualitative probabilistic system, and while a detailed discussion of synergy is beyond the scope of this article (see instead [6, 8, 30]), the following brief explanation is worthwhile.

The basic idea, in the language we are discussing here, is that the relationships between formulae are not completely modular in the same way that they are in logic. As an example, consider two implications $(i : a \rightarrow c : +)$ and $(i : b \rightarrow c : +)$. If these were logical implications, whatever was known about a would not affect the relationship between b and c. However, because we are dealing with probability, a change in what is known about a might change the relationship between b and c. For instance when the probability of a increases, this change may mean that $\Pr(c)$ increases less than before when $\Pr(b)$ increases[2]. It is this kind of interaction that synergy was first introduced [30] to capture, and the variety of synergy which describes this kind of interaction was later called *additive* synergy.

Additive synergy, however, is not sufficient to describe all the possible types of interaction between the causes of some formula. Consider the implications $(i : \mathit{sprinkler} \rightarrow \mathit{wet\_grass} : +)$ and $(i : \mathit{rain} \rightarrow \mathit{wet\_grass} : +)$ which capture the fact that both rain and the use of a sprinkler make it more likely that the grass of my lawn will be wet. Now, if I know that my grass is wet, then as Pearl [25]

[2]It should be noted that because the constraint on the probabilities of b and c is written in the way it is, taking into account all the possible other things that may affect $\Pr(c)$ in addition to $\Pr(b)$, it can never be the case that a change in $\Pr(a)$ will change the relationship between b and c to the extent that increasing $\Pr(b)$ leads to a decrease in $\Pr(c)$.

famously pointed out, if $\Pr(sprinkler)$ increases, then $\Pr(rain)$ decreases because the use of the sprinkler *explains away* the wet grass. This kind of *intercausal* [8] reasoning is described by another form of synergy—*product* synergy.

As mentioned above, in this paper synergies are represented by formulae such as $a \uplus b \rightsquigarrow c$ which represents the synergy which exists between a and b with respect to c. Such synergistic formulae form the basis of triples such as $(i : a \uplus b \rightsquigarrow c : +)$ in just the same way as simple and implicational formulae do, but with yet another denotation. In particular, $(i : a \uplus b \rightsquigarrow c : +)$ denotes the fact that:

$$\Pr(c|a, b, X).\Pr(c|\neg a, \neg b, X) \geq \Pr(c|\neg a, b, X).\Pr(c|a, \neg b, X)$$

where as before, X ranges across all other formulae such that there are triples $(i : x \rightarrow c : s)$. Similarly, $(i : a \uplus b \rightsquigarrow c : -)$ denotes the fact that:

$$\Pr(c|a, b, X).\Pr(c|\neg a, \neg b, X) \leq \Pr(c|\neg a, b, X).\Pr(c|a, \neg b, X)$$

and $(i : a \uplus b \rightsquigarrow c : 0)$ denotes the fact that:

$$\Pr(c|a, b, X).\Pr(c|\neg a, \neg b, X) = \Pr(c|\neg a, b, X).\Pr(c|a, \neg b, X)$$

In the terminology of [6] these are product synergies. In this paper we do not consider additive synergies, though they could be incorporated into the framework if it were desired, because they are of less direct use than product synergies. Furthermore we only consider synergies with values $+$, 0, and $-$ though categorical synergies are certainly conceivable.

3 The proof theory

The previous section introduced a language for describing probabilistic influences between formulae. For this to be useful we need to give a mechanism for taking sentences in that language and using them to derive new sentences. In particular we need to be able to take sentences describing changes in probability in particular formulae and use these, along with implicational and synergistic formulae to establish changes in probability in other formulae. This is done using the consequence relation $\vdash_{QP}$ which is defined in Figure 1. The definition is in terms of Gentzen-style proof rules where the antecedents are written above the line and the consequence is written below. The consequence relation operates on a database of the kind of triples introduced in the previous section and derives *arguments* about formulae from them. The concept of an argument is formally defined as follows:

Definition 1 *An* argument *for a well-formed formula p from a database Δ is a triple (p, G, s) such that $\Delta \vdash_{QP} (p, G, s)$*

The *sign* s of the argument denotes something about the probability of p while the *grounds* G identify the elements of the database used in the derivation of p.

C-rules

$$\text{Ax}\frac{}{\Delta \vdash_{QP} (St, \{i\}, Sg)}\ (i : St : Sg) \in \Delta$$

$$\wedge\text{-E1}\frac{\Delta \vdash_{QP} (St \wedge St', G, Sg)}{\Delta \vdash_{QP} (St, G, \mathsf{conj}_{\mathsf{elim}}(Sg))}$$

$$\wedge\text{-E2}\frac{\Delta \vdash_{QP} (St \wedge St', G, Sg)}{\Delta \vdash_{QP} (St', G, \mathsf{conj}_{\mathsf{elim}}(Sg))}$$

$$\wedge\text{-I}\frac{\Delta \vdash_{QP} (St, G, Sg) \quad \Delta \vdash_{QP} (St', G', Sg')}{\Delta \vdash_{QP} (St \wedge St', G \cup G', \mathsf{conj}_{\mathsf{intro}}(Sg, Sg'))}$$

$$\neg\text{-E}\frac{\Delta \vdash_{QP} (\neg St, G, Sg)}{\Delta \vdash_{QP} (St, G, \mathsf{neg}(Sg))}$$

$$\neg\text{-I}\frac{\Delta \vdash_{QP} (St, G, Sg)}{\Delta \vdash_{QP} (\neg St, G, \mathsf{neg}(Sg))}$$

$$\rightarrow\text{-E}\frac{\Delta \vdash_{QP} (St, G, Sg) \quad \Delta \vdash_{QP} (St \rightarrow St', G', Sg')}{\Delta \vdash_{QP} (St', G \cup G', \mathsf{imp}_{\mathsf{elim}}(Sg, Sg'))}$$

E-rules

$$\rightarrow\text{-R}\frac{\Delta \vdash_{QP} (St', G, Sg) \quad \Delta \vdash_{QP} (St \rightarrow St', G', Sg')}{\Delta \vdash_{QP} (St, G \cup G', \mathsf{imp}_{\mathsf{rev}}(Sg, Sg'))}$$

I-rules

$$\rightsquigarrow\text{-E1}\frac{\Delta \vdash_{QP} (St \uplus St' \rightsquigarrow St'', G, Sg) \quad \Delta \vdash_{QP} (St, G'', Sg') \quad \Delta \vdash_{QP} (St'', G''', 1)}{\Delta \vdash_{QP} (St', G \cup G' \cup G''', \mathsf{syn}_{\mathsf{elim}}(Sg, Sg'))}$$

$$\rightsquigarrow\text{-E2}\frac{\Delta \vdash_{QP} (St \uplus St' \rightsquigarrow St'', G, Sg) \quad \Delta \vdash_{QP} (St', G'', Sg') \quad \Delta \vdash_{QP} (St'', G''', 1)}{\Delta \vdash_{QP} (St, G \cup G' \cup G''', \mathsf{syn}_{\mathsf{elim}}(Sg, Sg'))}$$

Figure 1: The consequence relation $\vdash_{QP}$

To see how the idea of an argument fits in with the proof rules in Figure 1, consider the rules 'Ax', '$\wedge$-I' and '$\rightarrow$-E'. The first says that from a triple $(i : l : s)$ it is possible to build an argument for l which has sign s and a set of grounds $\{i\}$ (the grounds thus identify which elements from the database are used in the derivation). The rule is thus a kind of bootstrap mechanism to allow the elements of the database to be turned into arguments which other rules can

then be applied to. The second rule says that from arguments for two different formulae it is possible to build an argument for their conjunction. The set of grounds for this argument is the union of the grounds for the two individual arguments and the sign is a function of their signs. The rule '$\rightarrow$-E' can be thought of as analogous to modus ponens. From an argument for a and an argument for $a \rightarrow c$ it is possible to build an argument for c once the necessary book-keeping with grounds and signs has been carried out.

Example 1. Consider the following database which denotes the fact that the proposition "*premise*" has a probability which increases to 1, and that there is a relation between the proposition *premise* and the proposition "*conclusion*" such that if the probability of *premise* becomes 1, so does the probability of *conclusion*.

$$\begin{array}{ll} (f1 : premise : \Uparrow) & \Delta_1 \\ (r1 : premise \rightarrow conclusion : ++) & \end{array}$$

From the database, by application of Ax and $\rightarrow$-E, it is possible to build the argument:

$$\Delta_1 \vdash_{QP} (conclusion, \{r1, f1\}, \Uparrow)$$

since applying $\mathsf{imp}_{\mathsf{elim}}$ to $\Uparrow$ and $++$ yields $\Uparrow$ (as we will see in a little while). Thus from the database it is possible to build an argument for the probability of *conclusion* becoming 1. $\Box$

The proof procedure used here differs in a couple of important ways from other similar logical proof systems. Both of these differences stem from the fact that $\mathcal{QPR}$is dealing with probability values (albeit changes in probability) rather than just truth and falsity as is the case in classical logic. The first difference is that it matters whether there are several proofs for a given formula. In logic once there is a valid proof for a formula, the formula is known to be true. Here there may be an argument which suggests that the probability of a formula increases and another which suggests it decreases—to resolve the conflict it is necessary to combine the arguments as discussed later on. The second difference is that it is usual to have two sets of proof rules for each connective, one set which specify how to introduce the connective into formulae and one set which specify how to eliminate connectives from formulae. The proof rules in Figure 1 mainly consist of elimination rules. This reflects the focus of the system described in the paper which is intended to capture the reasoning possible in qualitative probabilistic networks. The system is thus intended to be used to establish changes in the probability of sets of formulae rather than to establish connections between sets of formulae—it is connections between sets of formulae, themselves formulae of the form $a \rightarrow c$, which are the kind of formulae that the proof rules cannot build. Were the missing introduction rules included we would have a system which was capable, in the language of probabilistic networks, of inferring new arcs connecting nodes in addition to inferring things about nodes.

In order to apply the proof rules to build arguments, it is necessary to supply the functions used in Figure 1 to combine signs. This section introduces those

	1	$\Uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\wr$
1	1	$\Uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\wr$
$\Uparrow$	$\Uparrow$	$\Uparrow$	$\uparrow$	$\uparrow$	$\updownarrow$	$\Downarrow$	0	$\updownarrow$	$\uparrow$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\updownarrow$	$\Downarrow$	0	$\updownarrow$	$\uparrow$
$\leftrightarrow$	$\leftrightarrow$	$\uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\leftrightarrow$
$\downarrow$	$\downarrow$	$\updownarrow$	$\updownarrow$	$\downarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\downarrow$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	0	$\Downarrow$	$\Downarrow$
0	0	0	0	0	0	0	0	0	0
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\Downarrow$	0	$\updownarrow$	$\updownarrow$
$\wr$	$\wr$	$\uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\wr$

Table 1: Conjunction introduction $\mathsf{conj}_{\mathsf{intro}}$ for conjuncts that are known to be independent.

functions and makes some remarks about the proof rules. The following section then proves the soundness and completeness of the proof procedure.

We start with conjunction introduction and elimination. When introducing conjunction it is crucially important whether the propositions in question are independent or not (since it is often not possible to establish the probability of the conjunction of a pair of dependent formulae from the probabilities of the formulae alone). If the formulae are known to be independent then the following definition applies.

Definition 2 *The function* $\mathsf{conj}_{\mathsf{intro}} : Sg \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \times Sg' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \mapsto Sg'' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\}$ *is specified by Table 1 where, as with all combinator tables in this paper, the first argument is taken from the first column and the second argument is taken from the first row.*

If the formulae are not known to be independent, then the following definition applies instead.

Definition 3 *The function* $\mathsf{conj}_{\mathsf{intro}} : Sg \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \times Sg' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \mapsto Sg'' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\}$ *is specified by Table 2.*

and it is clear that in most cases it is not possible to tell how the probability of the conjunction changes. Note that both Table 1 and Table 2 are written so that when the result of a combination could be either $\wr$ or a change in value (as when $\wr$ is combined with $\leftrightarrow$) the result given is always the change. This is because we always know that the value of a probability is $\wr$ so giving it as a result is less informative—it is only included to ensure that the functions used by $\vdash_{QP}$, in particular $\mathsf{imp}_{\mathsf{elim}}$ and $\mathsf{imp}_{\mathsf{rev}}$, are closed. The reason for bothering to have separate definitions for independent and non-independent conjuncts is that it is possible to identify independent formulae once arguments have been built, and doing so allows more precise inferences to be made (as is easily seen by comparing Tables 1 and 2).

	1	⇑	↑	↔	↓	⇓	0	↕	≀
1	1	⇑	↕	↕	↕	⇓	0	↕	↕
⇑	⇑	⇑	↕	↕	↕	⇓	0	↕	↑
↑	↕	↕	↕	↕	↕	⇓	0	↕	↕
↔	↕	↕	↕	↕	↕	⇓	0	↕	↕
↓	↕	↕	↕	↕	↕	⇓	0	↕	↕
⇓	⇓	⇓	⇓	⇓	⇓	⇓	0	⇓	⇓
0	0	0	0	0	0	0	0	0	0
↕	↕	↕	↕	↕	↕	⇓	0	↕	↕
≀	↕	↕	↕	↔	↕	⇓	0	↕	↕

Table 2: Conjunction introduction $\mathsf{conj}_{\mathsf{intro}}$ for conjuncts that are not known to be independent.

s	1	⇑	↑	↔	↓	⇓	0	↕	≀
s'	0	⇓	↓	↔	↑	⇑	1	↕	≀

Table 3: Negation of *swff*s neg

When eliminating a conjunction with sign Sg we assign both conjuncts the sign:

$$\mathsf{conj}_{\mathsf{elim}}(Sg) = \begin{cases} 1, \Uparrow & \text{if Sg} = \Uparrow \\ 1 & \text{if Sg} = 1 \\ \updownarrow & \text{otherwise} \end{cases}$$

where $\mathsf{conj}_{\mathsf{elim}}(Sg) = 1, \Uparrow$ means that $\mathsf{conj}_{\mathsf{elim}}(Sg)$ is either 1 or $\Uparrow$ (such values are propagated by carrying out the appropriate computation on both the signs in question [22]). What this means is that most of the time it is not possible to determine how the probability of the conjuncts change. This is an unfortunate but unavoidable property of probability theory and can be seen to follow from conjunction introduction—$\mathsf{conj}_{\mathsf{elim}}$ is just the inverse of $\mathsf{conj}_{\mathsf{intro}}$.

The rules for handling negation are applicable only to *swff*s and permit negation to be either introduced or eliminated by altering the sign, for example allowing $(i : \neg a : \uparrow)$ to be rewritten as $(i : a : \downarrow)$. This leads to the definition of neg:

Definition 4 *The function* $\mathsf{neg} : (i : \neg a : s), s \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \mapsto (i : a : s'), s' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\}$ *relates* s *to* s' *by Table 3.*

Note that neg is not defined over the values $++$, $+-$, $+$, 0, $-$, $-+$, and $--$. Although an implication $(i : a \rightarrow b : +)$ has a kind of inverse relation with $(i : a \rightarrow b : -)$, there is no such relation with $(i : \neg(a \rightarrow b) : s)$. Indeed, $(i : \neg(a \rightarrow b) : s)$ is not even an implication, since its main connective is $\neg$. It is not possible to apply neg to an implication—if neg is applicable, the formula it is applied to is not an implication. (In fact the alert reader will have noticed that $(i : \neg(a \rightarrow b) : s)$ is not even a well-formed formula.)

	$++$	$+-$	$+$	0	$-$	$-+$	$--$	$?$
1	1	$\wr$	$\wr$	$\wr$	$\wr$	$\wr$	0	$\wr$
$\Uparrow$	$\Uparrow$	$\uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\downarrow$	$\Downarrow$	$\updownarrow$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\updownarrow$
$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\leftrightarrow$	$\uparrow$	$\updownarrow$	$\uparrow$	$\updownarrow$
$\Downarrow$	$\downarrow$	$\Downarrow$	$\downarrow$	$\leftrightarrow$	$\uparrow$	$\Uparrow$	$\uparrow$	$\updownarrow$
0	$\wr$	0	$\wr$	$\wr$	$\wr$	1	$\wr$	$\wr$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\wr$	$\wr$	$\wr$	$\wr$	$\wr$	$\wr$	$\wr$	$\wr$	$\wr$

Table 4: Implication elimination $\mathsf{imp}_{\mathsf{elim}}$

To deal with implication we need two further functions, $\mathsf{imp}_{\mathsf{elim}}$ to establish the sign of formulae generated by the rule of inference $\rightarrow$-E, and $\mathsf{imp}_{\mathsf{rev}}$ to establish the sign of formulae generated by $\rightarrow$-R. This means that $\mathsf{imp}_{\mathsf{elim}}$ is used to combine the change in probability of a formula a, say, with the constraint that the probability of a imposes upon the probability of another formula c. Since this constraint is expressed in exactly the same way as qualitative influences are in QPNs, $\mathsf{imp}_{\mathsf{elim}}$ performs the same function as $\otimes$ [30], and is merely an extension of it.

Definition 5 *The function* $\mathsf{imp}_{\mathsf{elim}} : Sg \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \times Sg' \in \{++, +-, +, 0, -, -+, --, ?\} \mapsto Sg'' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\}$ *is specified by Table 4.*

There are two things that are notable about Table 4. First, the asymmetry in the table. This stems from the definition of the categorical implications. If the asymmetry did not exist, categorical implications would be close to logical bi-implications. Second, the fact that in this table, unlike those introduced previously, $\wr$ is the result of combining two signs neither of which is $\wr$, for instance 1 and $+$. This is the justification for including $\wr$ as a sign—if it were not included the set of signs would not be closed under $\mathsf{imp}_{\mathsf{elim}}$.

The function $\mathsf{imp}_{\mathsf{rev}}$ which computes the sign of the antecedent of an implication from that of the implication and its consequent, is similar, only differing in the way it handles categorical implications:

Definition 6 *The function* $\mathsf{imp}_{\mathsf{rev}} : Sg \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \times Sg' \in \{++, +-, +, 0, -, -+, --, ?\} \mapsto Sg'' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\}$ *is specified by Table 5.*

The difference is that categorical implications are only categorical in the direction in which they are specified. When reversed implications with signs $++$ and $+-$ behave in the same way as implications with sign $+$, and implications with signs $--$ and $-+$ behave in the same way as implications with sign $-$. Note that, once again, $\wr$ is required to ensure closure.

We also need the function $\mathsf{syn}_{\mathsf{elim}}$ in order to be able to reason with synergies. This is follows directly from [31]:

	++	+−	+	0	−	−+	−−	?
1	↯	↯	↯	↯	↯	↯	↯	↯
⇑	↑	↑	↑	↔	↓	↓	↓	↕
↑	↑	↑	↑	↔	↓	↓	↓	↕
↔	↔	↔	↔	↔	↔	↔	↔	↔
↓	↓	↓	↓	↔	↑	↕	↑	↕
⇓	↓	↓	↓	↔	↑	↑	↑	↕
0	↯	↯	↯	↯	↯	↯	↯	↯
↕	↕	↕	↕	↔	↕	↕	↕	↕
↯	↯	↯	↯	↯	↯	↯	↯	↯

Table 5: Implication reversal $\mathsf{imp}_{\mathsf{rev}}$.

	1	⇑	↑	↔	↓	⇓	0	↕
+	↯	↑	↑	↔	↓	↓	↯	↕
0	↔	↔	↔	↔	↔	↔	↔	↔
−	↯	↓	↓	↔	↑	↑	↯	↕

Table 6: Synergy elimination $\mathsf{syn}_{\mathsf{elim}}$.

Definition 7 *The function* $\mathsf{syn}_{\mathsf{elim}} : Sg \in \{+, 0, -\} \times Sg' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \lightning\} \mapsto Sg'' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \lightning\}$ *is specified by Table 6.*

The function $\mathsf{syn}_{\mathsf{elim}}$ is the last function required to define $\vdash_{QP}$, and we can turn to issues of soundness and completeness.

4 Soundness and completeness

Now, the aforementioned baroque appearance of the system might lead the sceptical reader to assume that the definitions given above are rather *ad hoc* and not to be trusted. However, they are not. The proof mechanism given above is provably sound and complete for the propagation of changes in probability in the sense that it only computes changes that will occur according to probability theory, and it computes all such changes. This is shown by the results in this section. However, the business of proving soundness and completeness is not straightforward. The main problem is that the form of the results depends heavily on the kind of reasoning. As a result we have three sets of soundness and completeness results. The first is for causal reasoning, that is reasoning in the direction of the implications only. The second is for evidential reasoning, that is reasoning both in the direction of implications and in the opposite direction to implications, and involves dealing with the problems of d-separation. The third is for intercausal reasoning, that is reasoning that includes the elimination of synergies.

4.1 Causal reasoning

As mentioned above, a restriction when writing implicational formulae in $\mathcal{QPR}$ is that the direction of the implications must reflect causality in the same way that the direction of a directed arc in a probabilistic network [25] does. That is the consequent of an implication must be an effect of the antecedent. The reason for insisting on this direction is exactly the same as in probabilistic networks—to ensure that reasonable conclusions are drawn. If we restrict the kinds of arguments we build to those in which implications are only used in a causal direction, in other words we only use the C-rules of $\vdash_{QP}$ to build arguments, soundness and completeness results are quite straightforward.

To show this we first need to define what it means for formulae to be causes and effects of one another:

Definition 8 *A well-formed formula p is said to be a cause of a well-formed formula q if and only if it is possible to identify an ordered set of* iwffs *$\{a_1 \rightarrow c_1, \ldots, a_n \rightarrow c_n\}$ such that q is one of the conjuncts that make up c_n or includes one or more of the conjuncts that make up c_n, one or more of the conjuncts in every a_i is also in c_{i-1}, and p is one of the conjuncts that make up a_1 or includes one or more of the conjuncts in a_1.*

In other words In other words p is a cause of q if it is possible to build up a trail of (causally directed) implications which link p to q.

Definition 9 *A well-formed formula p is said to be an effect of a well-formed formula q if and only if it is possible to identify an ordered set of* iwffs *$\{a_1 \rightarrow c_1, \ldots, a_n \rightarrow c_n\}$ such that q is one of the conjuncts that make up a_1 or includes one or more of the conjuncts that make up a_1, one or more of the conjuncts in every c_i is also in a_{i+1}, and p is one of the conjuncts that make up c_n or includes one of the conjuncts in c_n.*

Thus p is an effect of q if it is possible to build up a trail of (causally directed) implications that link q to p.

Given these definitions, it is possible to show that given information about the change in probability of some formula p, the C-rules of $\vdash_{QP}$ may be used to soundly and completely compute arguments about the changes in probability of the effects of p. However, this is not enough to show that $\mathcal{QPR}$ is sound and complete when using the C-rules. The problem is that, in general, there may be several different arguments for a single formula, and we therefore need a means of combining these in a sound way. We therefore define a *flattening* function flat which combines arguments by mapping from a set of arguments $\mathbf{A}$ to the supported formula p and some overall measure of validity:

$$\mathsf{flat} : \mathbf{A} \mapsto \langle p, v \rangle$$

where v is the result of a suitable combination of the signs of the arguments. Now, because the effect of each implication is defined to occur whatever other arguments are formed (this is a result of the constraint imposed on the conditional probabilities by the implications), all combinations are completely local,

	1	$\Uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\wr$
1	1	1	1	1	1			1	1
$\Uparrow$	1	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$			$\Uparrow$	$\Uparrow$
$\uparrow$	1	$\Uparrow$	$\uparrow$	$\uparrow$	$\updownarrow$	$\Downarrow$	0	$\updownarrow$	$\uparrow$
$\leftrightarrow$	1	$\Uparrow$	$\uparrow$	$\leftrightarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\leftrightarrow$
$\downarrow$	1	$\Uparrow$	$\updownarrow$	$\downarrow$	$\downarrow$	$\Downarrow$	0	$\updownarrow$	$\downarrow$
$\Downarrow$			$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	0	$\Downarrow$	$\Downarrow$
0			0	0	0	0	0	0	0
$\updownarrow$	1	$\Uparrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\Downarrow$	0	$\updownarrow$	$\updownarrow$
$\wr$	1	$\Uparrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\Downarrow$	0	$\updownarrow$	$\wr$

Table 7: Flattening flat

and the structure of the arguments may be disregarded when flattening[3] (for exactly the same reason as when combining evidential trails in QPNs [7]). As a result, v is simply calculated as:

$$v = \bigoplus_i Sg_i$$

for all $(p, G_i, Sg_i) \in \mathbf{A}$ where $\oplus$ is an extended version of the qualitative addition function used by QPNs, defined as follows:

Definition 10 *The function* $\oplus : Sg \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, 0, \updownarrow, \wr\} \times Sg' \in \{1, \Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, \updownarrow, \wr\} \mapsto Sg'' \in \{\Uparrow, \uparrow, \leftrightarrow, \downarrow, \Downarrow, \updownarrow, \wr\}$ *is specified by Table 7. Blank spaces represent impossible combinations.*

With this function established we can at last give the overall procedure for determining the change in a formula p in which we are interested. Assuming that all known changes in probability in formulae are in the database, this is:

1. Add a triple $(i : q : s)$ for every formula q whose change in probability is known.
2. Build $\mathbf{A}$, the set of all arguments for p using the C-rules.
3. Flatten this set to $\mathsf{Flat_c}(\mathbf{A})$ where $\mathsf{Flat_c}(\mathbf{A}) = \mathsf{flat}(\mathbf{A})$[4].

This naturally backward chaining procedure can obviously be extended to compute all the causal consequences of a given set of changes in probability.

From the definitions of how changes in probability are combined and then flattened, it is possible to show that using the C-rules of $\vdash_{QP}$ is sound:

Theorem 11 *The construction and flattening of arguments in $\mathcal{QPR}$ using the C-rules of $\vdash_{QP}$ is sound with respect to probability theory.*

[3] Though in other argumentation systems where structure is important, as when dealing with numerical probabilities for instance, structure can be taken into account when flattening.

[4] The reason for doing this should become clear in the next section.

Proof: The proof starts by proving the soundness of the combinator tables used when applying the C-rules of $\vdash_{QP}$:
(Conjunction introduction): Consider the probabilities $\Pr(a)$ and $\Pr(b)$ of the two formulae being conjoined. There are two cases to consider, in the first the formulae are known to be independent and in the second they are not known to be independent. If $\Pr(a)$ and $\Pr(b)$ are independent, then $\Pr(a \wedge b) = \Pr(a).\Pr(b)$. Thus if at least one of $\Pr(a)$ and $\Pr(b)$ increases and the other does not decrease, then $\Pr(a \wedge b)$ will increase. If one increases and one decreases, then the change in $\Pr(a \wedge b)$ cannot be determined. If one increases to 1, and the other is either 1 or increases to 1, $\Pr(a \wedge b)$ increases to 1. If both $\Pr(a)$ and $\Pr(b)$ are 1 then $\Pr(a \wedge b)$ is 1. Furthermore, if at least one of $\Pr(a)$ and $\Pr(b)$ decreases and the other does not increase, then $\Pr(a \wedge b)$ will decrease. If one decreases and one increases, then the change in $\Pr(a \wedge b)$ cannot be determined. If either $\Pr(a)$ or $\Pr(b)$ decrease to 0, then $\Pr(a \wedge b)$ decreases to 0, and if either is 0, then so is $\Pr(a \wedge b)$. If either a or b has the sign $\updownarrow$, then that is the sign of the conjunction, unless the other conjunct has a probability which is 0 or decreases to 0, since nothing can be said about its value. If one of a or b has the sign $\leftrightarrow$, and the other has a sign which is either 1, $\leftrightarrow$ or $\wr$ then the probability of the conjunction does not change and so the conjunction has the sign $\leftrightarrow$. If a or b has the sign $\wr$ and the other has the sign $\wr$ or 1, then the probability of the conjunction is not known, though it does not change and so we give the conjunction the sign $\wr$ (preferring this to $\leftrightarrow$ since it makes it clear that there has been no combination with a value which is known to be $\leftrightarrow$, though it would not be incorrect to use $\leftrightarrow$). This completes the proof for the case in which $\Pr(a)$ and $\Pr(b)$ are independent.

Turning to the case in which $\Pr(a)$ and $\Pr(b)$ are not independent, the same sets of values can be considered. However, since $\Pr(a \wedge b) = \Pr(a).\Pr(b|a) = \Pr(a|b).\Pr(b)$, it rapidly becomes clear that unless either $\Pr(a)$ or $\Pr(b)$ is 0 or decreases to 0, in which case $\Pr(a \wedge b)$ is zero and decreases to zero respectively, there is little that can be said about the probability of the conjunction because there is no constraint on the way in which $\Pr(b|a)$ and $\Pr(a|b)$ change. Indeed, the only time that the probability of the conjunction can be predicted is when both $\Pr(a)$ and $\Pr(b)$ are 1 or increase to 1. If both are 1 then so is $\Pr(a \wedge b)$, and if one increases to 1 and the other is either 1 or increases to 1, the $\Pr(a \wedge b)$ increases to 1. In all other cases, $\Pr(a \wedge b)$ will have the sign $\updownarrow$.
(Conjunction elimination): There are two parts to the proof. One for the part of the function that gives $\Uparrow$ or 1, and one for the part that gives $\updownarrow$. For the first, the following suffices and follows directly from the functions for conjunction introduction. The only way in which $Pr(a \wedge b)$ can be 1 is if both $\Pr(a)$ and $\Pr(b)$ are 1. The only way in which $Pr(a \wedge b)$ can increase to 1 is if either $\Pr(a)$ or $\Pr(b)$ increases to 1 and the other increases to 1 or is 1. Thus picking one of the two conjuncts, its probability either increases to 1 or is 1. Thus the first part is proved. For the second part we need the following argument. Giving any sign as $\updownarrow$ is always sound (since it means that nothing at all is being said about the relevant probability). However, it is also possible to prove that no more precise rule can be proposed. This is done by considering the tables for

conjunction introduction. Looking at the values in the tables it is clear that any sign that might be assigned to $\Pr(a \wedge b)$ might be produced by a number of possible values of $\Pr(a)$ and $\Pr(b)$. Thus no firm conclusions about changes in $\Pr(a)$ and $\Pr(b)$ can be drawn from particular changes in $\Pr(a \wedge b)$ other than $\Uparrow$ or 1.

(Implication elimination): First consider implications labelled with $+$. From the definition of such implications it is clear that combining any increase in probability (either $\uparrow$ or $\Uparrow$) with such an implication will generate a possible increase in probability, in other words a $\uparrow$. Similarly, combining any decrease in probability (either $\downarrow$ or $\Downarrow$) with an implication labelled $+$ will generate $\downarrow$, combining no change in probability with such an implication will generate $\leftrightarrow$, and combining a change of $\updownarrow$ with such an implication will generate a change of $\updownarrow$. Next, consider implications labelled with $-$. Such an implication will also generate $\updownarrow$ when combined with a change of $\updownarrow$ and $\leftrightarrow$ when combined with $\leftrightarrow$, but will otherwise have the opposite behaviour to that of an implication labelled $+$. It is also clear that implications labelled with 0 will generate changes of $\leftrightarrow$ whatever change they are combined with, and implications labelled with $\updownarrow$ will generate $\updownarrow$ when combined with all changes except $\leftrightarrow$—in the latter case they will generate a change of $\leftrightarrow$. Combining any of these implications with values that are not changes (that is 1, 0 or $\wr$) will not yield either a change or a value which is known, in other words they will generate $\wr$. This takes care of all non-categorical implications.

Turning to categorical implications, the results also follow almost immediately from the definitions. Consider an implication labelled $++$. By definition this yields a change of $\Uparrow$ when combined with a change of $\Uparrow$ and a value of 1 when combined with 1, and otherwise behaves exactly like an implication labelled with $+$. Similarly, by definition an implication labelled $+-$ will give $\Downarrow$ when combined with a change of $\Downarrow$ and 0 when combined with 0, but will otherwise behave like an implication labelled with $+$. The last two implications, $--$ and $-+$ behave in a complementary fashion to $++$ and $+-$ respectively.

(Negation elimination and introduction): Consider a, the formula whose sign is being computed. If $\Pr(\neg a)$ increases to 1 then clearly $\Pr(a)$ decreases to zero, and if $\Pr(\neg a)$ decreases to 0 then clearly $\Pr(a)$ increases to one. This takes care of the function for $\Downarrow$ and $\Uparrow$. The other cases are handled similarly.

This completes the proof of the soundness of the relevant combinator tables. Because the tables are sound, the arguments built using them are also sound. Thus all the arguments that may be built concerning a formula are sound. All that remains is to show that when several arguments for a formula are combined, the combination itself is sound. Thus it is necessary to show that flattening is sound.

(Flattening): The soundness of flattening follows from the fact that Table 7 is an extension of the qualitative addition function $\oplus$ used to combine changes in probability in qualitative probabilistic networks [30]. The differences between flat and $\oplus$ reflect the fact that categorical changes in probability cannot be altered by non-categorical changes and the spaces in the table follow from the fact that the probability of any variable cannot both increase to 1 and decrease

to 0 simultaneously [19]. Thus flattening is sound.
Since both building and flattening arguments is sound, $\mathcal{QPR}$ itself is sound. □

The notable thing about this result, in contrast to later ones, is that the soundness of the individual arguments generated by $\vdash_{QP}$ does not depend upon flattening. Each argument is itself sound—once we have an argument which says, for instance, that the probability of p may increase, that is we have an argument $(p, G, \uparrow)$, it is not possible to deduce that the probability of p won't increase. The closest that one can come to deducing that p won't increase is if it is possible to build another argument $(p : G' : \downarrow)$, which says that the probability of p might decrease. Then the result of flattening these two is the conclusion $\langle p, \updownarrow \rangle$ which indicates that it is not possible to rule out any change in p. The whole deduction has a certain locality which makes $\mathcal{QPR}$ rather closer to logic than to probabilistic networks. However, the fact that later conclusions can throw doubt on earlier ones means that $\mathcal{QPR}$ is not monotonic (a point explored at greater length in [23]).

Before moving on to completeness, we need to identify precisely what kind of completeness we are talking about. What we want to show is that when using the C-rules, $\mathcal{QPR}$ computes all the changes in probability of all the effects of a formula:

Definition 12 *The construction and flattening of arguments is said to be* causally complete *in some system of qualitative probability with respect to some formula p if it is possible to use that system to compute the sign of all the effects of p.*

With this definition it is possible to state and prove the following theorem.

Theorem 13 *The construction and flattening of arguments in $\mathcal{QPR}$ with $\vdash_{QP}$ defined by the C-rules only is causally complete with respect to any well-formed formula.*

Proof: Thanks to the careful choice of proof rules, the completeness proof follows from the definition of $\vdash_{QP}$. That is the change in probability of all the effects of any well-formed formula p which may be stated in $\mathcal{QPR}$ can be made by the application of the appropriate proof rules. This can be seen as follows. Consider the addition of the triple $(i : p : \uparrow)$ where p contains no negation symbols, to a database which only contains formulae without negation symbols. There are six types of effect of p. The first are the consequents of implications in which p forms the antecedent. The changes in probability of such effects may be established using $\rightarrow$-E. The second are the consequents of implications in which p is one of the conjuncts of the antecedent. The changes in probability of such effects may be established using $\wedge$-I and $\rightarrow$-E. The third are are the consequents of implications the antecedent of which involves some conjuncts that are part of p. The changes in probability of such effects may be established using $\wedge$-E1, $\wedge$-E2 and $\rightarrow$-E. The fourth set of effects are those which are subsets of the conjuncts in the consequents of implications which have p either as the

antecedent, one of the conjuncts of the antecedent, or for which some of the conjuncts that make up p are the antecedent. The changes in probability of such effects may be established by the following method. Use what is known about p to establish the change in probability of the whole consequent by applying the appropriate method (one of the first three) and then applying $\wedge$-E and $\wedge$-E2. A very similar procedure can be used to establish the change in probability of effects which include some conjuncts in the consequent of implications for which p relates to the antecdent, possibly using $\wedge$-I as well. Such formulae constitute the fifth set of effects. These five sets of effects are all those which are connected in some way to p by a single implication. The sixth set of effects are those which are related to p by two or more implications. The changes in probability of such sets may be obtained by recursively applying the procedure for the first five sets of effects. The appropriate use of $\neg$-I and $\neg$-E make it possible to formulae situations in which negation symbols appear. Thus all the changes in causes of p that result from the change in probability of p can be computed, and $\mathcal{QPR}$ is causally complete with respect to any formula. $\Box$

Example 2. As an example of causal reasoning consider the following example borrowed from [7]. We have the following probabilistic influences[5]:

$$\begin{array}{ll} (r1 : HeOx\_Temp \rightarrow Ox\_Tank\_Leak : +) & \Delta_2 \\ (r2 : HeOx\_Temp \rightarrow High\_Ox\_Temp : +) & \\ (r3 : High\_Ox\_Temp \rightarrow Ox\_Tank\_Leak : +) & \end{array}$$

When we have evidence that $\Pr(HeOx\_Temp)$ is increasing, so that the triple $(f1 : HeOx\_Temp : \uparrow)$ is added to the database, it is possible to build two arguments concerning $\Pr(Ox\_Tank\_Leak)$:

$$\begin{array}{lll} \Delta_2 & \vdash_{QP} & (Ox\_Tank\_Leak, \{f1, r1\}, \uparrow) \\ \Delta_2 & \vdash_{QP} & (Ox\_Tank\_Leak, \{f1, r1, r2\}, \uparrow) \end{array}$$

The first is built by combining $f1$ and $r1$ using $\rightarrow$-E. The second is built by combining $f1$ and $r2$ using $\rightarrow$-R and then chaining the result of this with $r3$ using $\rightarrow$-E again. These two arguments may then be flattened to give the pair $\langle Ox\_Tank\_Leak, \uparrow \rangle$. $\Box$

This is all we will say about causal reasoning using $\mathcal{QPR}$, and we turn to using the system to reason both in the direction of the implications, and in the opposite direction to the implications.

4.2 Evidential reasoning

Unfortunately there is more to allowing implications to be reversed than just adding the proof rule $\rightarrow$-R to $\vdash_{QP}$. In particular there are two problems which

[5]Of course the variables are binary valued rather than continuous as in the original so we must think of variable values such as $HeOx\_Temp = high$ rather than actual temperatures.

need to be solved. The first problem arises because when implications are reversed it is possible for the proof procedure to loop and therefore build an infinite number of arguments. This is illustrated by the following example.

Example 3. Consider the following database:

$$\begin{array}{ll} (f1 : a : \uparrow) & \Delta_3 \\ (r1 : a \rightarrow b : +) & \\ (r2 : b \rightarrow d : +) & \\ (r3 : a \rightarrow c : -) & \\ (r4 : c \rightarrow d : +) & \end{array}$$

By applying $\rightarrow$-E twice to $f1$, $r1$ and $r2$ it is possible to build an argument for an increase in the probability of d and then by using $\rightarrow$-R twice on $r4$ and $r3$ it is possible to build an argument for a decrease in the probability of a. This new information about a may then be used to build a new argument for a decrease in the probability of d, and this in turn can be used to build a new argument for an increase in the probability of a. This process could clearly be continued for ever. □

In fact, it is not even necessary to have a "loop" in the implications since it is perfectly possible to build a causal argument from a to b and then to d and then build an evidential argument back to b and then to a. Happily this problem is easy to solve by introducing the idea of a minimal argument.

Definition 14 *A* minimal argument *is an argument in which no implication appears more than once.*

The way that minimality is introduced in $\mathcal{QPR}$, as we shall see, is in the flattening of evidential arguments. This is conceptually simple since it allows the construction (as opposed to flattening) of arguments to be the same in both causal and evidential cases. However, there are a couple of points that should be made with reference to this. The first is that in practice it is both simple and desirable to check for minimality during the construction of arguments. Simple because it is easy to check whether an implication has been used before when applying the proof rules and desirable since it prevents the proof system being used to build infinite arguments. The second point is that under the usual restriction placed on probabilistic networks, cycles of implications (which would make it possible to reason causally from a formula and cycle back to it again) are forbidden in $\mathcal{QPR}$ so that non-minimal arguments are not a feature of causal reasoning. It is also worth noting that minimal arguments mirror the idea of minimal trails introduced by Druzdzel [7].

The second problem with evidential arguments arises due to the need to handle conditional independence properly. If we apply the proof rules blindly, we may build arguments concerning a formula which depend upon information about other formulae which are conditionally independent of it. Thus it is possible to build arguments which are not valid according to probability theory, and, just like the non-minimal arguments discussed above, they must be

weeded out in the flattening process. To identify invalid arguments we need to develop something for arguments in $\mathcal{QPR}$ which is analagous to d-separation [25] in probabilistic networks. We do this using the following definition of d-separation adapted from those in Jensen's recent book [14] (because I can't imagine bettering Jensen's motivation for d-separation, any reader who wants more information about what it is and why it is important is referred to [14], pages 7–14). First, however, we need some additional definitions:

Definition 15 *A* source *of an argument (p, G, s) is an* swff *from G.*

Thus a source of an argument is one of the simple formula which ground it, and form the head of a chain of implications.

Definition 16 *The* destination *of an argument (p, G, s) is p.*

Definition 17 *Two formulae p and q are* d-separated *if for all arguments which have p as a source and q as their destination, there is another formula r such that either:*

1. *p is a cause of r, r is a cause of q and the probability of r is 1 or 0; or*
2. *p and q are both causes of r and there is no argument (r, G', s') such that all the* swffs *in G' are effects of r.*

With these ideas fixed we can establish the idea of an invalid argument as one that is built without taking account of d-separation:

Definition 18 *An argument $A = (p, G, s)$ is* invalid *if all the sources of A are d-separated from p.*

Definition 19 *An argument $A = (p, G, s)$ is* valid *if it is not invalid.*

In other words there are two situations in which an argument is invalid. The first is if it involves a chain of implications through some formula which is known to be either true or false. The second is if it involves a chain of implications from the causes of some formula r to r and then back to further causes of r and there is no argument for r from any of its effects. This is illustrated by the following example.

Example 4. Consider the following database:

$$\begin{array}{ll} (f1 : a : \uparrow) & \Delta_4 \\ (r1 : a \rightarrow b : +) & \\ (r2 : b \rightarrow c : +) & \\ (r3 : e \rightarrow c : -) & \\ (r4 : c \rightarrow f : +) & \end{array}$$

By applying $\rightarrow$-E twice to $f1$, $r1$ and $r2$ it is possible to build an argument $(c, \{f1, r1, r2\}, \uparrow)$ for an increase in the probability of c. This argument is valid, but would be invalid if the triple $(f2 : b : 1)$ were also in Δ_4.

Now, consider extending the argument for an increase in probability of c by using $\rightarrow$-R on $r3$ and what was deduced about c to build an argument for a decrease in the probability of e. This second argument $(e, \{f1, r1, r2, r3\}, \downarrow)$ is invalid, but would be valid if the triple $(f3 : f : \uparrow)$ were in the database because it would then be possible to build a valid argument whose destination was c and whose grounds only included the effects of c. $\Box$

The idea of an invalid argument makes it possible to eliminate the kind of problems discussed by Pearl [24] in his exhortation for the use of causality in default reasoning without the need to distinguish between causal and evidential rules. Furthermore, it gives $\mathcal{QPR}$ the same kind of ability as symbolic causal networks [5] to ensure that changes in belief, expressed as probabilities, are consistent with ideas of causality without the need to associate a network with a set of logical clauses. Of course, the need to identify invalid arguments and rule them out means that, when used for evidential reasoning, $\mathcal{QPR}$ is no longer purely local in the way in which it is when used for causal reasoning. However, it is precisely this non-locality which makes it possible to ensure that adequate account is taken of d-separation without the need to have a graphical model as well as the logical clauses.

In keeping with the style of presentation adopted so far, we can think of applying the minimality and validity restrictions on arguments by applying a function $\mathsf{flat}_{\mathsf{e}_1}$ to the set of all arguments $\mathbf{A}$ for a formula p:

$$\mathsf{flat}_{\mathsf{e}_1} : \mathbf{A} \mapsto \{A \in \mathbf{A} \mid A \text{ is minimal and valid}\}$$

Now, if there are several minimal valid arguments for a given formula, we can combine these to get a single overall argument using a second flattening function $\mathsf{flat}_{\mathsf{e}_2}$. Like $\mathsf{flat}_{\mathsf{e}_2}$ this maps from a set of arguments $\mathbf{A}$ to the supported formula p and some overall measure of validity:

$$\mathsf{flat}_{\mathsf{e}_2} : \mathbf{A} \mapsto \langle p, v \rangle$$

where v is once again the result of a suitable combination of the signs of the arguments. In fact it turns out that v is computed in exactly the same way as for causal reasoning, so that the function $\mathsf{flat}_{\mathsf{e}_2}$ is exactly the same as flat. Thus the procedure for finding the sign of a formula p when reasoning both causally and evidentially is:

1. Add a triple $(i : q : s)$ for every formula q whose change in probability is known.
2. Build $\mathbf{A}$, the set of all arguments for p using the C-rules and E-rules.
3. Flatten this set to $\mathsf{Flat}_{\mathsf{e}}(\mathbf{A})$ where $\mathsf{Flat}_{\mathsf{e}}(\mathbf{A}) = \mathsf{flat}(\mathsf{flat}_{\mathsf{e}_1}(\mathbf{A}))$.

With this procedure in mind, we can prove the following.

Theorem 20 *The construction and flattening of arguments in $\mathcal{QPR}$ using the C-rules and E-rules of $\vdash_{QP}$ is sound with respect to probability theory.*

Proof: The proof proceeds by showing first that the individual proof rules are locally sound, in that given particular premises they generate the appropriate conclusions, and then showing that the flattening procedure rules ensures the soundness of whole arguments. The first stage is particularly easy since the soundness of the C-rules was proved in Theorem 11. We therefore need only to consider implication reversal.

(Implication reversal): The soundness of Table 5 can be proved as follows. Any implication $(i : a \rightarrow c : +)$ indicates a constraint $\Pr(c|a, x) \geq \Pr(c|\neg a, x)$. This constraint implies that $\Pr(a|c, y) \geq \Pr(a|\neg c, y)$ as proved by Wellman [30]. This can be considered as meaning that a consequence of the first implication is that there is another implication $(i : c \rightarrow a : +)$ (though this will not be causally directed). This second implication can then be combined with information about the change in probability of c to obtain the relevant column in the table just as for implication elimination in the proof of Theorem 11. Similar reasoning takes care of the cases for which the sign of the implication is $-$, 0 and ?. A categorical implication $(i : a \rightarrow c : ++)$ or $(i : a \rightarrow c : -+)$ is just a more extreme version of $(i : a \rightarrow c : +)$, and while it won't necessarily reverse to give a categorical implication, it will reverse just like $(i : a \rightarrow c : +)$. Similarly a categorical implication $(i : a \rightarrow c : --)$ or $(i : a \rightarrow c : +-)$ is just a more extreme version of $(i : a \rightarrow c : -)$, and while it won't necessarily reverse to give a categorical implication, it will reverse just like $(i : a \rightarrow c : -)$. This completes the proof of the soundness of implication reversal.

Now, this local procedure will sometimes be unsound, but only in the course of building an invalid argument (since the only unsound arguments which may be built are invalid), and such an argument will be rejected by the flattening function. In fact, strictly speaking, we don't actually need to worry about d-separation at all. The worst that could happen if we ignored it is that some formula whose probability cannot change, because it is d-separated from the only formula whose probability is known to change, has its change in probability computed as $\uparrow$ or $\downarrow$ (it cannot be $\Uparrow$ or $\Downarrow$ because categorical changes cannot result from the application of $\rightarrow$-R). Since $\uparrow$ and $\downarrow$ indicate either a change or no change this is not incorrect, but it is possibly misleading.

(Flattening): There are two aspects to the soundness of flattening. The first is the soundness of minimal valid arguments, and the second is the soundness of the way in which such arguments are combined. Both follow from the close correspondence between implications and arcs in qualitative probabilistic networks. The first is proven as follows. Minimal valid arguments correspond to minimal active trails in QPNs [7] and the soundness of the changes in probability that they identify follows from the soundness of the individual combinations proven above and the fact that non-valid, non-minimal arguments (where the calculation of changes is not sound) are removed. The second aspect of soundness may then be shown. The validity of combining different arguments also follows from the correspondence with evidential trails and the fact that Table 7 is an extension of the qualitative addition function $\oplus$ used to combine the results of such trails [7]. The differences between flat and $\oplus$ reflect the fact that categorical changes in probability cannot be altered by non-categorical changes and

the spaces in the table follow from the fact that the probability of any variable cannot both increase to 1 and decrease to 0 simultaneously [19]. Thus flattening is sound.
Since both building and flattening arguments is sound, $\mathcal{QPR}$ itself is sound. □

Given that evidential reasoning is sound, the next question is to what extent is it complete. We are interested in the following notion of completeness:

Definition 21 *The construction and and flattening of arguments is said to be* evidentially complete *in some system of qualitative probability with respect to some formula p if it is possible to use that system to compute the signs of all the effects of p, all the causes of p and all the causes and effects of all the causes and effects of p.*

With this definition it is possible to prove the following:

Theorem 22 *The construction and flattening of arguments in $\mathcal{QPR}$ is causally and evidentially complete with respect to any formula.*

Proof: Given information about the change in probability of any well-formed formula, by Theorem 13 it is possible to calculate the change in probability of any effect of that formula. Now, a procedure which is identical to that described in Theorem 13 but using $\rightarrow$-R as well as $\rightarrow$-E may be applied to establish the change in probability of any effect of any well-formed formula. Applying both procedures recursively as necessary suffices to ensure evidential completeness. □

Finally, we have an example of evidential reasoning in $\mathcal{QPR}$.

Example 5. As an example of the kind of reasoning possible in $\mathcal{QPR}$ consider the extension of the example of causal reasoning:

$$\begin{array}{ll} (r1 : HeOx\_Temp \rightarrow HeOx\_Temp\_Probe : +) & \Delta_5 \\ (r2 : HeOx\_Temp \rightarrow High\_Ox\_Temp : +) & \\ (r3 : HeOx\_Temp \rightarrow Ox\_Tank\_Leak : +) & \\ (r4 : High\_Ox\_Temp \rightarrow Ox\_Tank\_Leak : +) & \end{array}$$

When we have evidence that $\Pr(HeOx\_Temp\_Probe)$ is increasing, so that the triple $(f1 : HeOx\_Temp\_Probe : \uparrow)$ is added to the database, it is possible to build two minimal, valid arguments concerning $\Pr(Ox\_Tank\_Leak)$:

$$\begin{array}{lll} \Delta_5 & \vdash_{QP} & (Ox\_Tank\_Leak, \{f1, r1, r3\}, \uparrow) \\ \Delta_5 & \vdash_{QP} & (Ox\_Tank\_Leak, \{f1, r1, r2, r4\}, \uparrow) \end{array}$$

The first is built by combining $f1$ and $r1$ using $\rightarrow$-R and then combining the result of this with $r3$ using $\rightarrow$-E. The second is built by combining $f1$ and $r1$ using $\rightarrow$-R and then chaining the result of this with $r2$ and $r4$ using $\rightarrow$-E twice. These combine to give the pair $\langle Ox\_Tank\_Leak, \uparrow\rangle$ indicating that

overall it is possible to infer that knowledge about the increasing probability of *HeOx_Temp_Probe*, which is the kind of thing that can be observed, makes it possible to infer that the probability of *Ox_Tank_Leak* may increase, which is the kind of thing that would be useful to know in the context of this example. □

Using the C and E-rules, $\mathcal{QPR}$ captures Wellman's version of QPNs [30] up to the handling of additive synergy. The next section discusses how to extend $\mathcal{QPR}$ so that it handles intercausal reasoning. Doing so permits $\mathcal{QPR}$ to capture Druzdzel's [6] version of QPNs which don't deal with additive synergy but do employ intercausal reasoning.

4.3 Intercausal reasoning

In comparison to the extension to evidential reasoning, the extension of $\mathcal{QPR}$ to enable it to allow intercausal reasoning is relatively straightforward. Because of the way the synergy elimination rules $\rightsquigarrow$-E1 and $\rightsquigarrow$-E2 are defined, it is only ever possible to apply them validly. Thus, all that we have to do is to add the I-rules to the proof procedure and we can immediately obtain a sound and complete system. No new flattening function is required since the I-rules do not introduce new forms of invalid argument.

As ever, before showing soundness we need to state the complete proof procedure, and we do this in the familiar backward chaining way—the procedure for finding the sign of a formula p when reasoning causally, evidentially and intercasually is:

1. Add a triple $(i : q : s)$ for every formula q whose change in probability is known.
2. Build $\mathbf{A}$, the set of all arguments for p using the C-rules, E-rules and I-rules.
3. Flatten this set to $\mathsf{Flat_e}(\mathbf{A})$ where $\mathsf{Flat_e}(\mathbf{A}) = \mathsf{flat}(\mathsf{flat_{e_1}}(\mathbf{A}))$.

Theorem 23 *The construction and flattening of arguments in $\mathcal{QPR}$ using the C-rules, E-rules and I-rules of $\vdash_{QP}$ is sound with respect to probability theory.*

Proof: We already have Theorem 20 which shows that $\vdash_{QP}$ combined with $\mathsf{Flat_e}(\cdot)$ is sound when using the C-rules and the E-rules. Thus all we need to show is that synergy elimination is sound with respect to probability theory. Fortunately the soundness of synergy elimination follows directly from the definition of $\mathsf{syn_{elim}}$ and Druzdzel's results on intercausal reasoning, and so the use of $\mathcal{QPR}$ with the C-rules, E-rules and I-rules is sound. □

So proving soundness is relatively easy. Proving completeness, as ever, is dependent upon defining a notion of completeness, and to do this we need to capture the fact that one formula can be related intercausally with another. In fact we

need to express the idea that two formulae can be directly related by an intercausal link (when they share a common effect and so are the two antecedents of a synergistic *wff*) and may also be indirectly related (when they are related via a number of intermediate formulae some of which are synergistic *wff*s). The first idea is captured by the notion of intercausal connection, the second by the notion of intercausal relation:

Definition 24 *A well-formed formula p is said to be* intercausally connected *to a well-formed formula q if and only if there is a* ywff $p \uplus q \rightsquigarrow r$ *for some formula r.*

Definition 25 *A well-formed formula p is said to be* intercausally related *to a well-formed formula q if and only if it is possible to identify an ordered set of* ywff*s* $\{x_1 \uplus y_1 \rightsquigarrow z_1, \ldots x_n \uplus y_n \rightsquigarrow z_n\}$*, where there is an argument* $(z_i : G_i : 1)$ *for each z_i, and p is either a cause or effect of x_1, each y_i is a cause or an effect of each x_{i+1} and q is a cause or effect of y_n.*

In other words, two formulae are intercausally related if it is possible to build an argument which has one as its source and the other as its destination, and they are joined by a chain of implications and synergy relations. We then have:

Definition 26 *The construction and and flattening of arguments is said to be* intercausally complete *in some system of qualitative probability with respect to some well-formed formula p if in addition to being evidentially complete, it is possible to calculate all the changes in probability of all formulae that p is intercausally related to.*

With this definition it is possible to prove the following:

Theorem 27 *The construction and flattening of arguments in $\mathcal{QPR}$ is intercausally complete with respect to any formula.*

Proof: Again the proof follows almost immediately from the corresponding result for evidential reasoning. Starting from a known change in a proposition p, evidential completeness guarantees that we can find the changes in probability of all causes and effects of p and the causes and effects of those causes and effects. Synergy elimination then makes it possible to soundly establish any changes in probability of any formulae that are intercausally connected to any of the causes and effects of p. Once again the calculation of changes in probability of the causes and effects of the intercausally connected formulae is guaranteed by evidential completeness, and the recursive application of synergy elimination ensures intercausal completeness. □

This kind of completeness is the same as is possible in a probabilistic network. In a probabilistic network it is possible to calculate the probability of any node which is connected, via a set of nodes, to nodes about which evidence is obtained. In $\mathcal{QPR}$, it is possible to compute the change in probability of any formula which is "connected", in the sense of being a cause of or an effect of or intercausally related to, any formula for which the change in probability is known.

Finally, we give an example of intercausal reasoning in $\mathcal{QPR}$.

Example 6. As an example of the kind of reasoning possible in the full version of $\mathcal{QPR}$ consider this final extension of the running example:

$$\begin{array}{ll}
(r1 : HeOx_Temp \rightarrow HeOx_Temp_Probe : +) & \Delta_6 \\
(r2 : HeOx_Temp \rightarrow High_Ox_Temp : +) & \\
(r3 : HeOx_Temp \rightarrow Ox_Tank_Leak : +) & \\
(r4 : High_Ox_Temp \rightarrow Ox_Tank_Leak : +) & \\
(r5 : Ox_Tank_Leak \rightarrow Ox_Pressure_Low : -) & \\
(r6 : HeOx_Valve_Problem \rightarrow Ox_Pressure_Low : -) & \\
(r7 : Ox_Tank_Leak \uplus HeOx_Valve_Problem \rightsquigarrow Ox_Pressure_Low : -) &
\end{array}$$

When we have evidence that $\Pr(HeOx\_Temp\_Probe)$ is increasing and oxygen pressure is known to be low so that the formulae $(f1 : HeOx\_Temp\_Probe : \uparrow)$ and $(f2 : Ox\_Pressure\_Low : 1)$ are added to the database, as before it is possible to build two minimal, valid arguments concerning $\Pr(Ox\_Tank\_Leak)$:

$$\begin{array}{lll}
\Delta_6 & \vdash_{QP} & (Ox_Tank_Leak, \{f1, r1, r3\}, \uparrow) \\
\Delta_6 & \vdash_{QP} & (Ox_Tank_Leak, \{f1, r1, r2, r4\}, \uparrow)
\end{array}$$

Both of these may then be used along with $f2$, $r7$, and $\rightsquigarrow$-E1 to build arguments concerning $\Pr(HeOx\_Valve\_Problem)$:

$$\begin{array}{lll}
\Delta_6 & \vdash_{QP} & (HeOx_Valve_Problem, \{f1, f2, r1, r4, r7\}, \downarrow) \\
\Delta_6 & \vdash_{QP} & (HeOx_Valve_Problem, \{f1, f2, r1, r2, r3, r7\}, \downarrow)
\end{array}$$

which flatten to tell us that the probability of $HeOx\_Valve\_Problem$ may decrease. Thus the overall impact of the evidence is to suggest that it has become more likely that there is a leak in the oxygen tank and less likely that there is a problem with the helium/oxygen tank valve. □

With these results, $\mathcal{QPR}$ gives us a sound proof-theoretic means of computing changes in probability propagated in both causal and evidential directions as well as across intercausal links. Thus $\mathcal{QPR}$ captures Druzdzel's version of QPNs. What this means is that if we encode our probabilistic knowledge of the world by writing down any set of *swff*s, *iwff*s and *ywff*s we can then build arguments for and against formulae using $\vdash_{QP}$ and use these to identify the changes in probability of those formulae warranted by probability theory. If, after building arguments and flattening we have an pair (St, Sg) where St is any *wff* then Sg indicates the change in probability of St, indicating it increases to 1 if $Sg = \Uparrow$, decreases if $Sg = \downarrow$ and so on. If, on the other hand we have (St, Sg) where St is an *iwff* $St' \rightarrow St''$ then Sg indicates the constraint between $\Pr(St')$ and $\Pr(St'')$, and if St is a *ywff* then Sg indicates the constraint between the three constituent formulae. The full denotation of any pair (St, Sg) is given by Tables 8, 9 and 10. Since $\mathcal{QPR}$ is sound and complete any sign computed in this way will be correct, and if there is enough information to compute the sign, then it will be computed.

If	and	and	then
$St = w$	$Sg = 1$		$\Pr(w)_{final} = 1$
$St = w$	$Sg = \Uparrow$		$\Pr(w)_{final} = 1$
$St = w$	$Sg = \uparrow$	$\Pr(w)_{initial} = p$	$p \leq \Pr(w)_{final} \leq 1$
$St = w$	$Sg = \leftrightarrow$	$\Pr(w)_{initial} = p$	$\Pr(w)_{final} = p$
$St = w$	$Sg = \downarrow$	$\Pr(w)_{initial} = p$	$p \geq \Pr(w)_{final} = 0$
$St = w$	$Sg = \Downarrow$		$\Pr(w)_{final} = 0$
$St = w$	$Sg = 0$		$\Pr(w)_{final} = 0$
$St = w$	$Sg = \updownarrow$	$\Pr(w)_{initial} = p$	$0 \leq \Pr(w)_{final} \leq 1$
$St = w$	$Sg = \wr$		$0 \leq \Pr(w)_{final} \leq 1$

Table 8: What a derived formula means (part 1).

If	and	then
$St = v \rightarrow w$	$Sg = ++$	$\Pr(w\|v, x) = 1$
$St = v \rightarrow w$	$Sg = +-$	$\Pr(w\|\neg v, x) = 0$
$St = v \rightarrow w$	$Sg = +$	$\Pr(w\|v, x) \geq \Pr(w\|\neg v, x)$
$St = v \rightarrow w$	$Sg = 0$	$\Pr(w\|v, x) = \Pr(w\|\neg v, x)$
$St = v \rightarrow w$	$Sg = -$	$\Pr(w\|v, x) \leq \Pr(w\|\neg v, x)$
$St = v \rightarrow w$	$Sg = -+$	$\Pr(w\|\neg v, x) = 1$
$St = v \rightarrow w$	$Sg = --$	$\Pr(w\|v, x) = 0$
$St = v \rightarrow w$	$Sg = ?$	The relationship between $\Pr(w\|v, x)$ and $\Pr(w\|\neg v, x)$ is unknown.

Table 9: What a derived formula means (part 2).

5 Discussion

The first question that arises when considering $\mathcal{QPR}$ is why $\mathcal{QPR}$ is better than the QPN formalism, and so worth developing. There are a couple of reasons why I think that this is so. Firstly, the system has the potential to be considerably more expressive than QPNs. As it stands, $\mathcal{QPR}$ can reason about conjunctions which QPNs can't, so it is more expressive (though it is arguable how useful the conjunctions are) and it is possible to extend $\mathcal{QPR}$ to handle disjunction and thus material implication [21] which makes it possible to combine logical deduction with the kind of probabilistic propagation discussed in this paper. Secondly, $\mathcal{QPR}$ has the potential to be a first order system and so could be used as a means of building specific QPNs from more general knowledge—a form of model-based knowledge construction. Thirdly, $\mathcal{QPR}$ seems to offer a more natural means of representing the kind of qualitative probabilistic information discussed here than QPNs do. The key to both QPNs and $\mathcal{QPR}$ is that the influences that they deal with are defined to hold irrespective of what other influences also hold. In other words the information contained in an *iwff* or an arc in a QPN is essentially modular and unaffected by whatever other influences

If	and	then
$St = u \uplus v \rightsquigarrow w$	$Sg = +$	$\Pr(w\|u,v,X).\Pr(w\|\neg u,\neg v,X))$ $\geq \Pr(w\|u,\neg v,X).\Pr(w\|\neg u,v,X)$
$St = u \uplus v \rightsquigarrow w$	$Sg = 0$	$\Pr(w\|u,v,X).\Pr(w\|\neg u,\neg v,X))$ $= \Pr(w\|u,\neg v,X).\Pr(w\|\neg u,v,X)$
$St = u \uplus v \rightsquigarrow w$	$Sg = -$	$\Pr(w\|u,v,X).\Pr(w\|\neg u,\neg v,X))$ $\leq \Pr(w\|u,\neg v,X).\Pr(w\|\neg u,v,X)$

Table 10: What a derived formula means (part 3).

exist in a particular model. This is reflected more directly in $\mathcal{QPR}$ than in QPNs since $\mathcal{QPR}$ only takes the structure of the influences into account when necessary (which is when d-separation comes into play).

Another question that might be posed is how $\mathcal{QPR}$ relates to Neufeld's probabilistic default reasoner [17, 18]. The answer seems to be that because the "rules" in $\mathcal{QPR}$ make stronger assertions than those in Neufeld's system, it is possible to get completeness results in $\mathcal{QPR}$ which are not possible in Neufeld's work. As an example, consider the way in which both systems represent the fact that c is positively influenced by both a and b. Both can conclude that c becomes more probable if a becomes more probable and that c becomes more probable if b becomes more probable. In $\mathcal{QPR}$ it is also possible to conclude that c becomes more probable if both a and b become more probable because the effects of a and b are defined to occur whatever other influences bear on c. However, in Neufeld's system if both a and b become more probable, nothing can be said about the change in probability of c. The relationship between Neufeld's system and the kind of proof theoretic reasoning provided by $\mathcal{QPR}$ is discussed further in [20, 21]. Of course, the flipside of this completeness is the need to make stronger assertions when writing down rules, and this will lead to more influences being given the sign ? because it is not possible to state that they hold whatever other information is true. Thus one can think of $\mathcal{QPR}$ as being limited to expressing precise assertions about less of the world than Neufeld's system but as a result being able to be more complete in the inferences it makes about the portion of the world it represents.

6 Summary

This paper has discussed a means of building a proof theoretic system which is capable of reasoning about changes in probability. It is thus in some senses an extension of previous work on systems of argumentation and of systems of qualitative probability. With a solid basis in probability theory, the system can be used to combine the advantages of a sound means of handling uncertainty with the flexibility of a logical method of knowledge representation [1], a flexibility that can be increased by extending it to a full first order system and including disjunction and material implication. Because of its qualitative nature, the

system may be used when probabilistic knowledge of a domain is incomplete, making it applicable to a wider range of situations than systems that depend on complete probabilistic information, while the fact that it is soundly based on probability theory make it a useful basis for a qualitative decision theory [11, 12]. The system described in this paper clearly has similarities with other systems described in the literature. Some of these similarities have been described in the paper. Others are explored elsewhere [20, 21]. Yet others, including those with the systems described in [3, 28, 29], are the subject of on-going work.

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