QUALITATIVE PROBABILITY AND ORDER OF MAGNITUDE REASONING

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In recent years there has been a spate of papers describing systems for probabilistic reasoning which do not use numerical probabilities. In some cases these systems are unable to make any useful inferences because they deal with changes in probability at too high a level of abstraction. This paper discusses one of the problems this level of abstraction can cause, and shows how the use of a technique for order of magnitude reasoning can reduce its impact.

Keywords: Qualitative probability, order of magnitude reasoning.

1. Introduction

In the past few years there has been a good deal of interest in qualitative approaches to reasoning under uncertainty—approaches which do not make use of precise numerical values. Qualitative abstractions of probabilistic networks[1,2], in particular, have proved popular, finding use in areas in which the full numerical formalism is either not necessary or not appropriate. Applications have been reported in explanation[3], diagnosis[4,5], engineering design[6], and planning[2].

Whereas in most probabilistic systems the main goal is to establish what the probabilities of hypotheses are when particular observations are made, in qualitative probabilistic networks[2] (QPNs) the main aim is to establish how values change. Since the approach is qualitative, the size of the changes are not the focus. It only matters whether a given change is positive, written as [+], negative [−], or zero [0]. Clearly this information is rather weak, but as the applications show it is sufficient for some tasks. Furthermore, reasoning with qualitative probabilities is much more efficient than reasoning with precise probabilities, since computation is quadratic in the size of the network[1], rather than NP-hard[7].

One of the stumbling blocks in applying QPNs more widely is that there are situations in which it is not possible to resolve the changes that they handle with any precision. In such cases the value of the change remains unknown, and it is written as [?]. Such values tend to multiply, reducing the useful conclusions which

$\otimes$	[+]	[0]	[−]	[?]
[+]	[+]	[0]	[−]	[?]
[0]	[0]	[0]	[0]	[0]
[−]	[−]	[0]	[+]	[?]
[?]	[?]	[0]	[?]	[?]

$\oplus$	[+]	[0]	[−]	[?]
[+]	[+]	[+]	[?]	[?]
[0]	[+]	[0]	[−]	[?]
[−]	[?]	[−]	[−]	[?]
[?]	[?]	[?]	[?]	[?]

Table 1. Sign multiplication and addition.

can be obtained from a QPN. The aim of this paper is to investigate how techniques from order of magnitude reasoning, in particular Dague's system ROM[K][8] can be used to reduce the proliferation of such [?] values.

2. Qualitative certainty networks

It is possible[9,10] to generalize the approach provided by qualitative probabilistic networks to what are termed qualitative certainty networks (QCNs). Using this approach it is possible to propagate qualitative probability, possibility[11,12] and evidence theory[13,14] in a uniform way.

QCNs are built around the notion of influences between variables, where the influence may be given a probabilistic semantics, as in QPNs, or a semantics in terms of possibility or Dempster-Shafer theory. Formally, a QCN is a pair $G = (V, Q)$, where V is a set of variables or nodes in the graph, represented by a capital letter, and Q is a set of sets of qualitative relations among the values of the variables which reflect the influences between the variables. In this paper we concentrate upon QCNs in which the influences, like the influences in QPNs, have a probabilistic semantics. These are known as probabilistic QCNs (QP/CNs).

The qualitative relations are expressed in terms of the derivatives that relate the different values of the variables together. If A has possible values $\{a_1, a_2, a_3\}$ and C has possible values $\{c_1, c_2\}$, then the relationship between the probability of a_1 and the probability of c_1 is specified by the derivative: $\partial \Pr(c_1)/\partial \Pr(a_1)$ thus the qualitative relationship between the probability of a_1 and the probability of c_1 is specified by $[\partial \Pr(c_1)/\partial \Pr(a_1)]$ where the square brackets denote that it is the qualitative value of the quantity that we are interested in. This means that we only take note of whether it is positive, which we denote by [+], negative, which we denote by [−] or is zero, which we denote by [0]. Then, if we write the qualitative change in the probability of A taking value a_1 as $[\Delta \Pr(a_1)]$ we have*:

$$[\Delta \Pr(c_1)] = \left[\frac{\partial \Pr(c_1)}{\partial \Pr(a_1)} \right] \otimes [\Delta \Pr(a_1)] \qquad (1)$$

which allows us to propagate changes in probability across influences between variables. All of this begs the question of how we determine what the qualitative influence between variables is, and it turns out[9] that:

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*Note that while this expression is correct for qualitative values, it is a linear approximation for exact numerical values.
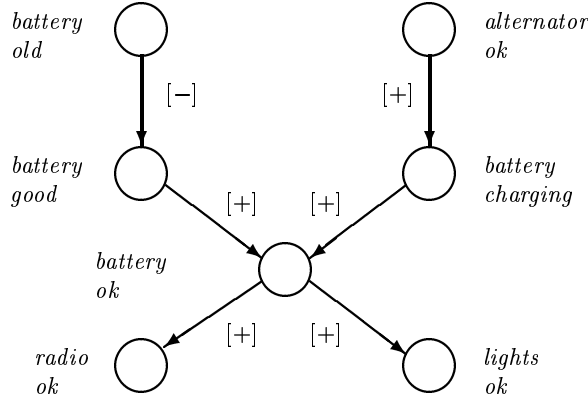
Fig. 2. Part of a car diagnosis network

Theorem 1 *The qualitative derivative:*

$$\left[\frac{\partial \Pr(c_1)}{\partial \Pr(a_1)}\right]$$

relating the probability of C taking value c_1 to the probability of A taking value a_1 has the value $[+]$, if, for all a_2 and X:

$$\Pr(c_1 \,|\, a_1, X) > \Pr(c_1 \,|\, a_2, X)$$

Derivatives with values $[-]$ and $[0]$ are obtained by replacing $>$ with $<$ and $=$. If a derivative cannot be determined to be $[+]$, $[-]$, or $[0]$, then it takes the value $[?]$. QCNs with possibilistic or Dempster-Shafer belief semantics handle changes in value in a similar way[9].

The impact of evidence on a given node can be calculated by taking the sign of the change in value at the evidence node and multiplying it by the sign of every link in the sequence that connects it to the node of interest. To see how this works, consider the example[5] in Figure 2 in which the value labeling each arc is the value of the qualitative derivative linking the probabilities of the events represented by the nodes at the end of the arc. If we observe that the radio is dead, so that the probability of the radio being ok decreases, $[\Delta \Pr(radio\ ok)] = [-]$, and we want to know the impact of this on the probability of the battery being good we calculate the effect as $[-] \otimes [+] \otimes [+]$. With the definition of sign multiplication $\otimes$ in Table 1 this gives a change in $\Pr(battery\ good)$ of $[-]$. If we also observed that the lights were not ok, and wanted to assess the impact of both pieces of evidence on the probability that the battery was good, we would establish the two individual effects and sum them using $\oplus$ (Table 1).

Described in these terms, QP/CNs are essentially equivalent to QPNs, the only difference being that the relation between two variables is described by a single qualitative value in a QPN and by a set of qualitative values in a QP/CN. However, QP/CNs can go somewhat further. In particular, we can describe the propagation

of values in terms of "separable" derivatives[9] where the effect of a change in the probability of one value of A on the probability of a value of C is calculated without considering its effects on the other values of A. We denote the qualitative separable qualitative derivative relating $\Pr(c_1)$ and $\Pr(a_1)$ by:

$$\left[\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}\right]$$

Previously[9] little use has been made of qualitative separable derivatives since their value is always [+], but in this paper they are key. The reason for this is that if we look at the quantitative value of separable derivative, we find that:

Theorem 2 *The separable derivative:*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}$$

relating the probability of C taking value c_1 to the probability of A taking value a_1, without taking other $\Pr(a_i)$, $i \neq 1$ into account, has the value $\Pr(c_1 \,|\, a_1)$.

The theorem follows directly from the value of the qualitative separable derivative relating the two quantities[9]. We also have[†]:

$$\Delta \Pr(c_1) = \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) + \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) + \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3) \quad (2)$$

3. Over-abstraction

The degree of abstraction in both QPNs and QCNs leads to situations in which certain changes may only be determined as [?] despite the presence of information that allows more precise inferences to be made. One way in which this can occur is when a [+] and a [−] are combined using ⊕, and several authors have investiagted ways to tackle this problem, which is known as "tradeoff resolution"[15,16,17]. A separate problem, which we will call "over-abstraction", is that for a broad class of networks there are values of C for which it is not possible to predict the effect of a change in the probability of a given value of A using Theorem 1 because the values of the conditional probabilities are such that the derivative which links the two has value [?]. This problem has addressed by Renooij and van der Gaag[18], for the situation of over-abstraction in QPNs, essentially by reasoning about individual values in the same way as in a QCN. The remainder of this paper looks at an alternative approach to handling over-abstraction which is equally applicable to QCNs and QPNs and goes somewhat further than Renooij and van der Gaag.

In some of these cases, it is possible to resolve this over-abstraction by using order-of-magnitude reasoning about the values of separable derivatives. As an ex-

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[†]Note that this expression is exact, unlike the case for the partial derivatives

ample[‡], consider a link from C to A in which it is known that:

$$\Pr(c_1 \,|\, a_1) \ll \Pr(c_1 \,|\, a_2) \qquad \Pr(c_1 \,|\, a_3) \approx \Pr(c_1 \,|\, a_2)$$

where $\ll$ indicates a difference of at least an order of magnitude. Information about the prior values is also available:

$$\Pr(a_3) \approx \Pr(a_2) \qquad \Pr(a_3) \ll \Pr(a_1) \qquad \Pr(a_1) \approx 1$$

In this situation applying Theorem 1 gives:

$$\left[\frac{\partial \Pr(c_1)}{\partial \Pr(a_2)}\right] = [?] \quad \left[\frac{\partial \Pr(c_1)}{\partial \Pr(a_3)}\right] = [?] \quad \left[\frac{\partial \Pr(c_1)}{\partial \Pr(a_1)}\right] = [-]$$

When we apply (1), we find that if there is an increase in $\Pr(a_1)$ then $[\Delta \Pr(c_1)] = [-]$ but if $\Pr(a_2)$ or $\Pr(a_3)$ increases, then $[\Delta \Pr(c_1)] = [?]$.

These ambiguous inferences can be resolved in some situations by using order of magnitude about the separable derivatives. From Theorem 2 we know that:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} \ll \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} \qquad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} \approx \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}$$

Consider that we find out that $\Pr(a_2)$ has become 1, meaning that $\Pr(a_1)$ and $\Pr(a_3)$ have become 0. Bearing the priors in mind:

$$|\Delta \Pr(a_3)| \ll |\Delta \Pr(a_2)| \approx |\Delta \Pr(a_1)|$$

Thus when we compare the magnitudes of the terms in (2) to establish the change in line fault probability for a delayed alarm, the second term dominates and we have $\Delta \Pr(c_1) = [+]$.

Handling this kind of reasoning formally is precisely what order of magnitude systems are intended to do, and the rest of this paper is concerned with formalizing variations of the above argument using one particular system of order of magnitude reasoning.

4. Order of magnitude reasoning

There are a number of systems which have been proposed for formal order of magnitude reasoning[21]. Of the initial proposals for order of magnitude reasoning, perhaps the most intuitively appealing is Raiman's system FOG[22] which makes it possible to represent and reason with information such as "Q_1 is negligible with respect to Q_2" and "Q_1 is of the same order of magnitude as Q_2". This style of reasoning was later refined by Dague in his system ROM[K][8]. ROM[K] gives four possible ways of expressing a relation between quantities: Q_1 is *negligible wrt* Q_2, $Q_1 \ll Q_2$, Q_1 is *distant from* Q_2, $Q_1 \not\sim Q_2$, Q_1 is *comparable to* Q_2, $Q_1 \sim Q_2$, and

[‡]This example is taken from previous work on a diagnosis system for electricity distribution networks[19,20].

(A1)	$A \approx A$
(A2)	$A \approx B \to B \approx A$
(A3)	$A \approx B, B \approx C \to A \approx C$
(A4)	$A \sim B \to B \sim A$
(A5)	$A \sim B, B \sim C \to A \sim C$
(A6)	$A \approx B \to A \sim B$
(A7)	$A \approx B \to C.A \approx C.B$
(A8)	$A \sim B \to C.A \sim C.B$
(A9)	$A \sim 1 \to [A] = [+]$
(A10)	$A \ll B \leftrightarrow B \approx (B + A)$
(A11)	$A \ll B, B \sim C \to A \ll C$
(A12)	$A \approx B, [C] = [A] \to (A + C) \approx (B + C)$
(A13)	$A \sim B, [C] = [A] \to (A + C) \sim (B + C)$
(A14)	$A \sim (A + A)$
(A15)	$A \not\sim B \leftrightarrow (A - B) \sim A$ or $(B - A) \sim B$
(P3)	$A \ll B \to C.A \ll C.B$
(P11)	$A \ll B, B \ll C \to A \ll C$
(P35)	$A \not\sim B \to C.A \not\sim C.B$
(P38)	$A \not\sim B, C \approx A, D \approx B \to C \not\sim D$

Table 3. Some of the axioms and properties of ROM[K].

Q_1 is *close to* Q_2, $Q_1 \approx Q_2$. We also write $Q_1 \gg Q_2$ to indicate $Q_2 \ll Q_1$. Once the relation between pairs of quantities is specified, it is possible to deduce new relations by applying the axioms and properties of ROM[K].

It should be noted that ROM[K] is a general scheme for carrying out order of magnitude reasoning, and the set of axioms in Table 3 are a minimal set which capture the properties of the set of relations. (The table also contains some of the properties which may be derived from these axioms, and which are used in this paper.) However, because of this generality, it is perfectly possible to use ROM[K] to reason about probability values. Indeed it has already been applied to tradeoff resolution[16].

4.1. *A procedure for resolving overabstraction*

The first step in the application of order of magnitude techniques is to obtain a result for any variation of the problem we started with. Such a result will make it possible to calculate the sign of any qualitative change in a probability $\Pr(c_1)$ given:

1. order of magnitude information about the conditionals $\Pr(c_1 \,|\, a_j)$ which relate it to the node A which influences it; and

2. order of magnitude information about changes in the values of the $\Pr(a_j)$.

Thus, given initial information:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} \text{ rel}_1 \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} \text{ rel}_2 \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} \tag{3}$$

	$\approx$	$\sim$	$\nsim$	$\ll$	$\gg$
$\approx$	$\approx$	$\sim$	$\nsim$	$\ll$	$\gg$
$\sim$	$\sim$	$\sim$	U	$\ll$	$\gg$
$\nsim$	$\nsim$	U	U	U	U
$\ll$	$\ll$	$\ll$	U	$\ll$	U
$\gg$	$\gg$	$\gg$	U	U	$\gg$

$\mathrm{rel_b}$ labels the columns; $\mathrm{rel_a}$ labels the rows.

Table 4. How to establish $\mathrm{rel_c}$ (Theorem 3).

$$\Delta \Pr(a_1) \ \mathrm{rel_3} \ \Delta \Pr(a_2) \ \mathrm{rel_4} \ \Delta \Pr(a_3) \tag{4}$$

where $\mathrm{rel_i} \in \{\approx, \sim, \nsim, \ll, \gg\}$, we can use the following procedure. Note that throughout this procedure we are only interested in the absolute values of quantities since the signs are taken into account by the fact that we are looking to determine the overall sign of:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) - \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) + \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

in other words the case in which $\Pr(a_1)$ and $\Pr(a_3)$ increase, and $\Pr(a_2)$ decreases. The results of all other cases involving three values can be established from this case by symmetry.

Step 1 Establish the relations between the products of separable derivative and change:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) \quad \mathrm{rel_5} \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) \quad \mathrm{rel_6} \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

using the following result:

Theorem 3 *Given*

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)} \ \mathrm{rel_a} \ \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}$$

and

$$\Delta \Pr(y) \ \mathrm{rel_b} \ \Delta \Pr(z)$$

where $\mathrm{rel_a}, \mathrm{rel_b} \in \{\approx, \sim, \nsim, \ll, \gg\}$, *then the relation* $\mathrm{rel_c}$ *that holds between:*

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(y)$$

and

$$\frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z)$$

is given by Table 4. U indicates that the relation may not be established.

$$\text{rel}_e$$

	$\approx$	$\sim$	$\not\approx$	$\ll$	$\gg$
$\approx$	$\gg$	$\not\approx$	$\sim, \ll^*$	$\ll$	$\approx$
$\sim$	$\gg$	U	$\sim, \ll^*$	$\ll$	$\sim$
$\not\approx$	$\gg, U^\dagger$	$\gg, U^\dagger$	$\ll, \gg^\ddagger$	$\sim, \ll^\dagger$	$\not\approx$
$\ll$	U	U	$\ll$	$\ll$	$\ll$
$\gg$	$\gg$	$\gg$	$\gg, \sim^*$	$\sim$	$\gg$

(with row label rel_d)

Table 5. How to establish rel_f (Theorem 4).

Step 2 From the result of the first step, establish the relationship between one product and the absolute value of the difference between the others:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} . \Delta \Pr(a_1) \ \text{rel}_7 \left(\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2) - \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} . \Delta \Pr(a_3) \right)$$

using Theorem 4.

Theorem 4 *Given:*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} . \Delta \Pr(a_1) \qquad \text{rel}_d \qquad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2)$$

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2) \qquad \text{rel}_e \qquad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} . \Delta \Pr(a_3)$$

then the relation rel_f *such that:*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} . \Delta \Pr(a_1) \ \text{rel}_f \left(\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2) - \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} . \Delta \Pr(a_3) \right)$$

is given by Table 5 where $*$ *indicates that the first relation holds if:*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} . \Delta \Pr(a_3)$$

and the second holds otherwise, $\dagger$ *indicates that the first relation holds if:*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} . \Delta \Pr(a_1) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2)$$

and the second holds otherwise, and $\ddagger$ *indicates that the first relation holds if* [§]:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} . \Delta \Pr(a_1) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2)$$

and the second holds if

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} . \Delta \Pr(a_1) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)} . \Delta \Pr(a_2) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} . \Delta \Pr(a_3)$$

otherwise no relation can be established. U *indicates that the relation may not be established*

[§]Since all the relations are $\not\approx$, we don't have to worry about the case in which the quantities are equal.

Step 3 From the result of the previous step, establish the sign of:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) - \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) + \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

using Theorem 5.

Theorem 5 *Given:*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) \quad \mathrm{rel}_g \quad \left(\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) - \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3) \right)$$

the sign of:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) - \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) + \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

is [+] *if* rel_g *is* $\gg$ *or if* rel_g *is* $\ll$ *and*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) \leq \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

The sign is [−] *if* rel_g *is* $\ll$ *and*

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

Otherwise the sign is [?].

This three step process makes it possible to determine the sign of the change at a three-valued node from relative order of magnitude information. Of course this only works for the case in which $\Delta \Pr(a_1)$ and $\Delta \Pr(a_3)$ are in one direction and $\Delta \Pr(a_2)$ is in the opposite direction. If this is not the case, we will require initial information other than that in (3) and (4). For instance, if $\Delta \Pr(a_3)$ decreases while and $\Delta \Pr(a_1)$ and $\Delta \Pr(a_2)$ increase, then we will need to know:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)} \quad \mathrm{rel}_1 \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)} \quad \mathrm{rel}_2 \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}$$

and

$$\Delta \Pr(a_1) \; \mathrm{rel}_3 \; \Delta \Pr(a_3) \; \mathrm{rel}_4 \; \Delta \Pr(a_2)$$

rather than (3) and (4) to apply the procedure.

4.2. *Handling more than three values*

Clearly it would be useful to have a method for finding the change at nodes with more than three values, and it turns out that such a method may be obtained by applying Theorem 4 recursively. Consider the extension of the case we have been

dealing with to that in which A has four possible values $a_1, \ldots a_4$. There are thus four changes in probability and four separable derivatives:

$$\Delta \Pr(a_1), \ldots, \Delta \Pr(a_4) \qquad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}, \ldots, \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_4)}$$

to take into account. We can, as before, apply Theorem 3 to obtain the relative orders of magnitude of the products of change and derivative such as:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) \quad \text{rel}_1 \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) \tag{5}$$

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) \quad \text{rel}_2 \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3) \tag{6}$$

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_4)}.\Delta \Pr(a_4) \quad \text{rel}_3 \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) \tag{7}$$

Then, writing:

$$X \quad \text{for} \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) \qquad\qquad Y \quad \text{for} \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

$$Z \quad \text{for} \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3) \qquad\qquad W \quad \text{for} \quad \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_4)}.\Delta \Pr(a_4)$$

Theorem 4 can be applied to give us:

$$X \quad \text{rel}_4 \quad Y - Z \qquad\qquad W \quad \text{rel}_5 \quad Y - Z \tag{8}$$

reversing the relation in the last equation, and applying Theorem 4 again will give us X rel$_6$ $Y - (Z + W)$ from which Theorem 5 will give us: $[X - Y + Z + W]$. If instead we require the sign of one product minus the other three, we need to use Theorem 4 to give W rel$_{5'}$ $Z - Y$ instead of (8). The former can then be re-written as $-W$ rel$_{5''}$ $Y - Z$ and Theorem 4 applied again to give: X rel$_6$ $Y - (Z - W)$ from which from which Theorem 5 will give us: $[X - Y + Z - W]$.

Thus the problem for four values of A is solved. Clearly Theorem 4 could be applied again to allow us to handle five or more possible values of A, and so we have a general procedure. Equally clearly, in order to apply this procedure we need to have a specific set of relations between changes and derivatives—in other words if we did not have the information about the relative magnitudes of products in (5)−(7) we would not be able to obtain the relationship between X and $Y - (W + Z)$. However, I don't think that this is unreasonable. What we have is a method for inferring those order of magnitude relations which follow from what is known. If some relationship cannot be established from what is known, then it follows that the available information is insufficient to allow conclusions to be drawn.

5. Discussion

The previous section has shown how ROM[K] can be used to resolve overabstraction in qualitative probabilistic reasoning. Like the only previous work on

this topic—Renooij and van der Gaag's use of "provoking variables" to resolve nonmonotonic influences[18]—the work presented here does not resolve the overabstraction once and for all. Instead, the method works for a specific set of changes in probability, and so the resolution of overabstraction must be carried out every time that probabilities are propagated. This obviously adds to the computational complexity of propagation algorithms for QPNs[1] and QCNs[9].

However, this increase in complexity need not be great. The propagation algorithm for QP/CNs[9] has a time complexity that is linear in the number of nodes in the network and quadratic in the number of values of the variables represented by the nodes, and a space complexity which is quadratic in both. The algorithm tracks both the qualitative values of the changes in probability at each node and the derivatives that correspond to arcs between nodes, and the datastructures which hold information could easily be extended to hold the relative orders of magnitude of the changes and derivatives. Given this information, it is computationally simple to carry out the procedure given in the previous section, since that procedure can be reduced to a look-up. Adding this computation to the standard propagation algorithm will increase the time complexity of computations at a given node, but will not affect the overall linear complexity in the number of nodes.[¶]

Of course, the procedure discussed here will not completely solve the problem of overabstraction. Indeed, all it will do is to provide a means of resolving it in some specific cases. However, as in all applications of qualitative reasoning, given the fact that reducing overabstration is a very hard problem to solve, any method which helps to reduce the number of [?] values that are generated has a useful role to play. Furthermore the method introduced here has the advantage of being applicable in a wider range of situations than previous work on the topic[18]—Renooij and van der Gaag's approach, while similar in spirit, would not be able to handle the example we have been discussing.

6. Summary and future work

The main results of this paper are to show that order of magnitude reasoning can be used to resolve overabstraction in qualitative probabilitistic reasoning, and to give formal results that allow this resolution. This work is far from being the final word on the subject, but does go further in resolving overabstraction than any similar work. There are two points which should be made about the method presented in this paper.

First, the generality of the QCN framework means that the results can be ap-

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[¶]The linear time complexity of the existing QP/CN algorithm comes from the fact that each node can change value just twice, from an initial value of [0] to [+] or [−] and then to [?]. Given n nodes, each with m values, if each node changes value twice then there are $2mn$ computations, each involving m multiplications and m additions[9]. The method described in this paper augments the additions and multiplications with look-ups, and the number of look-ups required will grow exponentially in m as we have to iterate applications of Theorem 4. Thus the approach will be linear in n and exponential in m. This makes the method tractable provided that the number of values of the variables modelled in the underlying network is comparatively small.

plied to resolve indeterminate values when qualitative versions of possibility and Dempster-Shafer theories are used—there is nothing in the technique which makes it specific to probability. Second, it should be noted that the method is heuristic. As with other order of magnitude techniques, there is a trade-off between drawing safe conclusions which are correct but unhelpful and drawing more aggressive conclusions which are more useful but which can be wrong. In the case of the technique employed here, the trade-off emerges from the mapping from numerical values to ROM[K] relations. The more aggressive the mapping—the more small relative differences are mapped in to $\ll$, $\gg$ and $\not\sim$ relations—the more the ambiguity that can be resolved, but also the larger the chance of an error. Conversely, the more that the mappings are made safe—the more that large relative differences are mapped into $\approx$ and $\sim$ relations—the less the ambiguity can be resolved, but the safer the conclusions are guaranteed to be. The third point is related to this. When the approach concludes that the change is [?] it does not represent a failure, but the conclusion that it is not safe to make any more precise inference about the change.

Second, there is an obvious direction in which this work could usefully be extended, leading on from the observation that in this paper the information which is taken as input to the system is provided directly in order of magnitude terms, as ROM[K] relations. While this seems reasonable for some sets of quantities, it avoids the question of how one gets the relations in the first place—what mappings from numbers to relations are appropriate? Providing maximally safe mappings is the goal of future research, and seem likely to make use of Dague's system ROM[$\Re$][23] which permits numerical order of magnitude reasoning.

Acknowledgments

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Appendix: Proofs of theorems

Proof of Theorem 3: To establish the relative magnitude of the products we proceed on a case-by-case basis, starting at the top lefthand corner of Table and working across, bearing in mind that the combination is symmetric with respect to the diagonal from top left to bottom right, and that all results involving $\gg$ may be obtained by symmetry from those for $\ll$.

(i) For $\partial_s \Pr(x)/\partial_s \Pr(y) \approx \partial_s \Pr(w)/\partial_s \Pr(z)$ and $\Delta \Pr(y) \approx \Delta \Pr(z)$, we first apply A7 to each of the initial expressions to get:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(y) \approx \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(y) \qquad \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(y) \approx \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z)$$

then we apply A3 to these two to get:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(y) \approx \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z)$$

which is the required result.

(ii) For $\partial_s \Pr(x)/\partial_s \Pr(y) \approx \partial_s \Pr(w)/\partial_s \Pr(z)$ and $\Delta \Pr(y) \sim \Delta \Pr(z)$, we apply broadly the same procedure as in (i), using A6 to get $\sim$ from $\approx$.

(iii) For $\partial_s \Pr(x)/\partial_s \Pr(y) \approx \partial_s \Pr(w)/\partial_s \Pr(z)$ and $\Delta \Pr(y) \not\approx \Delta \Pr(z)$, we apply P35 to get:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(y) \not\approx \frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(z)$$

while as before A7 gives:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(z) \approx \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z)$$

since A1 tells us that:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(y) \approx \frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(y)$$

and A2 that:

$$\frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z) \approx \frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(z)$$

we can then apply P38 to these last three expressions to find that rel_c is $\not\approx$.

(iv) For $\partial_s \Pr(x)/\partial_s \Pr(y) \approx \partial_s \Pr(w)/\partial_s \Pr(z)$ and $\Delta \Pr(y) \ll \Delta \Pr(z)$, we first apply A7, as usual, to get:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(z) \approx \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z)$$

and then A6 to get:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta \Pr(z) \sim \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta \Pr(z)$$

Next we use P3 to obtain:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta\Pr(y) \ll \frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta\Pr(z)$$

and combining these latter two expressions using A11, we get the necessary result.
(v) For $\partial_s\Pr(x)/\partial_s\Pr(y) \sim \partial_s\Pr(w)/\partial_s\Pr(z)$ and $\Delta\Pr(y) \sim \Delta\Pr(z)$, we again proceed as in (i).
(vi) For $\partial_s\Pr(x)/\partial_s\Pr(y) \sim \partial_s\Pr(w)/\partial_s\Pr(z)$ and $\Delta\Pr(y) \not\sim \Delta\Pr(z)$, we can apply P35 in the same way that we usually apply A7, A8 and P11 to obtain:

$$\frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(y) \not\sim \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(z)$$

and A8 to obtain:
$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta\Pr(y) \sim \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(y)$$

However, ROM[K] deliberately does not allow $\sim$ to be combined with $\not\sim$ so that we cannot apply the usual method to establish a relation between the products. Because of this we cannot obtain a result for this case.
(vii) For $\partial_s\Pr(x)/\partial_s\Pr(y) \sim \partial_s\Pr(w)/\partial_s\Pr(z)$ and $\Delta\Pr(y) \ll \Delta\Pr(z)$, we proceed as in (iv).
(viii) For $\partial_s\Pr(x)/\Pr(y) \not\sim \partial_s\Pr(w)/\partial_s\Pr(z)$ and $\Delta\Pr(y) \not\sim \Delta\Pr(z)$, we can apply P35 in the same way that we usually apply A7, A8 and P11 to obtain:

$$\frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(y) \not\sim \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(z) \qquad \frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta\Pr(y) \not\sim \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(y)$$

However, $\not\sim$ is deliberately not transitive so that we cannot apply the usual method to establish a relation between the products. Because of this we cannot obtain a result for this case.
(ix) For $\partial_s\Pr(x)\Pr(y) \not\sim \partial_s\Pr(w)/\partial_s\Pr(z)$ and $\Delta\Pr(y) \ll \Delta\Pr(z)$, we have much the same problem as in the previous case in that we cannot chain $\not\sim$ with $\ll$.
(x) For $\partial_s\Pr(x)\Pr(y) \ll \partial_s\Pr(w)/\partial_s\Pr(z)$ and $\Delta\Pr(y) \ll \Delta\Pr(z)$, we apply P3 twice to obtain:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta\Pr(y) \ll \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(y) \qquad \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(y) \ll \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(z)$$

and then P11 to obtain:

$$\frac{\partial_s \Pr(x)}{\partial_s \Pr(y)}.\Delta\Pr(y) \ll \frac{\partial_s \Pr(w)}{\partial_s \Pr(z)}.\Delta\Pr(z)$$

as required. All other results follow by symmetry. $\square$

Proof of Theorem 4 There are two ways of proving this theorem. One is to proceed at the object level of ROM[K], in the same way that the previous proofs was

obtained, using results such as to establish relationships between one product and the difference of the others. A somewhat shorter proof can be obtained by reasoning at the meta-level—this is what we provide here. Here, there are 25 separate cases.

(i) rel_d is $\approx$ and rel_e is $\approx$. The result of the subtraction may be positive or negative, but its absolute value will be negligible with respect to $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$. Thus rel_f is $\gg$.

(ii) rel_d is $\approx$ and rel_e is $\sim$. The absolute value of the subtraction will be quite a lot smaller than the minuend. Thus rel_f is $\not\simeq$.

(iii) rel_d is $\approx$ and rel_e is $\not\simeq$. If

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

then the result of the subtraction is positive, and comparable to the value of the minuend. Since the minuend is close in value to $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$, the correct relation between the latter and the result of the subtraction is $\sim$. If the condition does not hold, then the result of the subtraction is negative and negligible with respect to that of the minuend. Thus the relation between the absolute value of the result of the subtraction and $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ is $\ll$.

(iv) rel_d is $\approx$ and rel_e is $\ll$. The result of the subtraction is negative and of almost the same absolute value as $\partial_s \Pr(c_1)/\partial_s \Pr(a_3).\Delta \Pr(a_3)$. Thus, $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ is negligible with respect to this absolute value and rel_f is $\ll$.

(v) rel_d is $\approx$ and rel_e is $\gg$. Since rel_e is $\gg$, the subtraction will have negligible effect on $\partial_s \Pr(c_1)/\partial_s \Pr(a_2).\Delta \Pr(a_32)$, and so rel_f will just be $\approx$.

(vi) rel_d is $\sim$ and rel_e is $\approx$. The absolute value of result of the subtraction will be negligible with respect to the absolute value of either quantity involved in the subtraction. Since the value of $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ is close to that of $\partial_s \Pr(c_1)/\partial_s \Pr(a_2).\Delta \Pr(a_2)$, it will be much greater than the absolute value of the result of the subtraction, and rel_f will therefore be $\gg$.

(vii) rel_d is $\sim$ and rel_e is $\sim$. A similar argument to that in (ii) applies, but because rel_d is $\sim$ rather than $\approx$ we can draw no conclusions about rel_f.

(viii) rel_d is $\sim$ and rel_e is $\not\simeq$. A similar argument to that in (iii) applies, and the result is the same.

(ix) rel_d is $\sim$ and rel_e is $\ll$. A similar argument to that in (iv) gives the conclusion that rel_f is $\ll$.

(x) rel_d is $\sim$ and rel_e is $\gg$. A similar argument to that in (v) gives the conclusion that rel_f is $\sim$.

(xi) rel_d is $\not\simeq$ and rel_e is $\approx$. The absolute value of the subtraction is going to be negligible with respect to the minuend and thus negligible with respect to the absolute value of $\partial_s \Pr(c_1)/\partial_s \Pr(a_2).\Delta \Pr(a_2)$. Thus rel_f is $\gg$ so long as:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

If not, then $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ might be of comparable size to the result of the subtraction and no conclusion can be drawn about rel_f.

(xii) rel_d is $\not\sim$ and rel_e is $\sim$. A similar argument to that in (xi) applies giving the same result.

(xiii) rel_d is $\not\sim$ and rel_e is $\not\sim$. If

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3) \qquad (9)$$

the result of the subtraction is negative and has an absolute value much larger than the minuend. If, in addition,

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

we know that $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ will be negligible with respect to the absolute value of the result of the subtraction, and can conclude that rel_f is $\ll$. If the second condition does not hold, then we cannot establish rel_f. If (9) does not hold then the result of the subtraction is positive and comparable to the value of the minuend. Since the minuend is far in value from $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$, then if

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

we can conclude that rel_f is $\gg$, while if this inequality is reversed, then rel_f is $\ll$.

(xiv) rel_d is $\not\sim$ and rel_e is $\ll$. The result of the subtraction is negative and much bigger than the absolute value of $\partial_s \Pr(c_1)/\partial_s \Pr(a_2).\Delta \Pr(a_2)$. If:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

Then the $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ will have the a similar absolute value to the result of the subtraction and rel_f will be $\sim$. If:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_1)}.\Delta \Pr(a_1) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2)$$

then $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$ will be negligible with respect to the result of the subtraction, and rel_f will be $\not\sim$.

(xv) rel_d is $\not\sim$ and rel_e is $\gg$. A similar argument to that for (v) means that rel_f is $\not\sim$.

(xvi) rel_d is $\ll$ and rel_e is $\approx$. This time the result of the subtraction is negligible with respect to $\partial_s \Pr(c_1)/\partial_s \Pr(a_2).\Delta \Pr(a_2)$, and so it is impossible to say what rel_f is.

(xvii) rel_d is $\ll$ and rel_e is $\sim$. A similar argument to that in (xvi) applies, and it is impossible to say what rel_f is.

(xviii) rel_d is $\ll$ and rel_e is $\not\sim$. If:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

then the result of the subtraction is positive and comparable to the value of the minuend. Since $(\partial_s \Pr(c_1)/\partial_s \Pr(a_1)).\Delta \Pr(a_1)$ is negligible with respect to the minuend, rel_f is $\ll$. If, on the other hand:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

then the result of the subtraction is negative and much larger than the minuend. Since we are only interested in the absolute value, this means that rel_f is $\ll$.

(xix) rel_d is $\ll$ and rel_e is $\ll$. This time the result of the subtraction is negative, and its absolute value is negligible with respect to $\partial_s \Pr(c_1)/\partial_s \Pr(a_3).\Delta \Pr(a_3)$. Thus rel_f is $\ll$.

(xx) rel_d is $\ll$ and rel_e is $\gg$. A similar argument to that for (v) means that rel_f is $\ll$.

(xxi) rel_d is $\gg$ and rel_e is $\approx$. The result of the subtraction will have an absolute value negligible with respect to the minuend which itself is negligible with respect to $\partial_s \Pr(c_1)/\partial_s \Pr(a_1).\Delta \Pr(a_1)$, and so rel_f is $\gg$

(xxii) rel_d is $\gg$ and rel_e is $\sim$. A similar argument to that in (xxi) applies.

(xxiii) rel_d is $\gg$ and rel_e is $\not\approx$. Reasoning as in (xviii), if:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) > \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

then the result of the subtraction is positive and comparable to the value of the minuend. Since the minuend is negligible with respect to $(\partial_s \Pr(c_1)/\partial_s \Pr(a_1))$. $\Delta \Pr(a_1)$ rel_f is $\gg$. If, on the other hand:

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

then the result of the subtraction is negative and much larger than the minuend. Since we are only interested in the absolute value, this means that rel_f is $\sim$.

(xxiv) rel_d is $\gg$ and rel_e is $\ll$. The result of the subtraction will be negative and have almost the same absolute value as the subtrahend. Thus $\partial_s \Pr(c_1)/\partial_s \Pr(a_2)$. $\Delta \Pr(a_2)$ will be negligible with respect both to the result and $\partial_s \Pr(c_1)/\partial_s \Pr(a_1)$. $\Delta \Pr(a_1)$, and the two latter quantities will be comparable. Thus rel_f is $\sim$.

(xxv) rel_d is $\gg$ and rel_e is $\gg$. A similar argument to that for (v) means that rel_f is $\ll$.

Proof of Theorem 5: If rel_g is $\gg$ then the result follows immediately. If rel_g is $\ll$ then the sign is $[+]$ provided that that

$$\frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_2)}.\Delta \Pr(a_2) < \frac{\partial_s \Pr(c_1)}{\partial_s \Pr(a_3)}.\Delta \Pr(a_3)$$

so that the bracketed term is positive. If not, then the sign is $[-]$. In all other cases the result is too close to predict and the sign is $[?]$. $\square$