A rough set approach to reasoning under uncertainty

Simon Parsons
Department of Electronic Engineering, Queen Mary and Westfield College, Mile End Road, London, E1 4NS, United Kingdom.

and

Advanced Computation Laboratory, Imperial Cancer Research Fund, Lincoln's Inn Fields, London WC2A 3PX, United Kingdom.

Miroslav Kubat
Computer Centre, Technical University of Brno, Udolni 19, 60200 Brno, Czechoslovakia.

Mirko Dohnal
Advanced Computation Laboratory, Imperial Cancer Research Fund, Lincoln's Inn Fields, London WC2A 3PX, United Kingdom.

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Abstract

Reasoning with uncertain information is a problem of key importance when dealing with information about the real world. Obtaining the precise numbers required by many uncertainty handling formalisms can be a problem. The theory of rough sets makes it possible to handle uncertainty without the need for precise numbers, and so has some advantages in such situations. This paper presents an introduction to various forms of reasoning under uncertainty that are based on rough sets. In particular, a number of sets of numerical and symbolic truth values which may be used to augment propositional logic are developed, and a semantics for these values is provided based upon the notion of possible worlds. Methods of combining the truth values are developed so that they may be propagated when augmented logic formulæ are combined, and their use is demonstrated in theorem proving.

1. Introduction

Any system designed to reason about the real world must be capable of dealing with uncertain information, that is information whose certainty may not be completely established, and incomplete knowledge about its domain. This is a direct consequence of the complexity of the real world and the finite size of the knowledge base that such a system has at its disposal. Uncertain information is often represented by attaching some numerical estimate to the facts in question. This numerical information is then propagated according to the axioms of some theory which seeks to guarantee that the final degree of certainty accorded to the answer to a query is exactly that determined by the degrees of the relevant facts in the knowledge base. This seems a natural approach to the problem, but there is a wealth of evidence, of which Kahneman *et al* . (1982) is a good example, to support the view that human beings, who after all compile the knowledge bases that contain the uncertain information, are not very good at dealing with numbers. There are also many papers, including (Dawes 1982) and (Chard 1991), which demonstrate that simple symbolic techniques are as good, in many cases, at dealing with uncertainty as complex numerical ones. However, the provision of a symbolic theory for reasoning under uncertainty is a problem that has not been widely considered. In an attempt to rectify this situation, this paper concentrates on symbolic methods on the grounds that a symbolic formalism can combine ease of use for a human expert with a suitable ability to handle uncertainty. The paper does, however, also deal with a related numerical measure.

1.2 Overview

Section 2.0 introduces the basic concepts of rough set theory, including the key notions of core and envelope, and discusses the use of rough sets in knowledge representation. Section 3.0 shows how the theory of rough sets may be extended to cope with computing the core and envelope of logical combinations of objects. Following from this, Section 4.0 introduces a family of symbolic and numerical quantifiers for propositional logic, based on various interpretations of rough sets, suitable for a wide range of reasoning tasks under uncertainty including theorem proving. Finally, Section 5.0 deals with the semantics of knowledge representation based on rough sets, providing an account of how truth values may be established for rough quantifiers. Throughout the paper there is a running comparison of the method for reasoning with rough sets and approaches based upon the theory of evidence [Shafer 1976], since there in much in common between them.

2. Rough set theory

Rough sets, originally introduced by Pawlak (1982), have been further developed and applied to a number of problems by Pawlak (1984), Orłowska and Pawlak (1984), Farinas del Cerro and Orłowska (1985), Pawlak *et al.* (1986), Wong *et al.* (1986) and Pawlak *et al.* (1988). This section discusses the basic ideas behind the theory before relating them to knowledge representation using logic and the handling of uncertainty.

2.1 Basic concepts of rough sets

Consider a set of elementary concepts or attributes $\mathbf{A} = \{A_1,...,A_n\}$ such as {green, blue, egg, ball, cube}. These concepts are the language which is available to describe the set of objects $\mathbf{E} = \{E_1,...,E_m\}$ that are being manipulated. For instance, $\mathbf{E}$ may be the set {green cube, blue cube, blue ball, egg}. Now, if the description of the E_j is based upon the A_i alone, it could well be the case that some of the E_j are indistinguishable since the values that distinguish them are not present in $\mathbf{A}$. For instance using the set of concepts {green, blue, egg, ball, cube} it is not possible to tell a blue prism from a blue cone since there is nothing in the set of descriptive terms to distinguish prism from cone.

Thus the use of a finite set $\mathbf{A}$ implies the existence of an equivalence relation $\approx$ such that $E_j \approx E_k$ and E_j and E_k cannot be distinguished from one another for a given $\mathbf{A}$, if for every i, A_i is an attribute of E_j if and only if it is an attribute of E_k. Thus there is a partition on $\mathbf{E}$:

$$\mathbf{P} = \{P_1,...,P_r\} \quad \text{where} \quad \bigcup P_i = \mathbf{E} \quad \text{and} \quad P_i \cap P_j = \emptyset \quad \text{for } i \neq j = 1,...,r \quad (1)$$

and each P_i is an equivalence class. Thus in the example, since $\mathbf{A}$ = {green, blue, egg, ball, cube} and $\mathbf{E}$ = {green cube, blue cube, blue ball, egg} it follows that P = {{green cube}, {blue cube, blue ball}, {egg}}. Let $T \in \mathbf{T} = \{T_1,...,T_p\}$ be an object, whose attributes are $T_A \subseteq \mathbf{A}$, that must be described in terms of the partitioned set of attributes E_i. The following definitions may be made:

$$T^c(\mathbf{P}, \mathbf{E}) = \{e: e \in P_i, P_i \subset T\} \quad (2)$$

$$T^e(\mathbf{P}, \mathbf{E}) = \{e: e \in P_i, P_i \cap T \neq \emptyset\} \quad (3)$$

where $T^c(\mathbf{P}, \mathbf{E})$ is the core of T based on $\mathbf{E}$ and $\mathbf{P}$, the set of all equivalent objects in $\mathbf{E}$ all of whose attributes are possessed by T, and $T^e(\mathbf{P}, \mathbf{E})$ is the envelope of T based on $\mathbf{E}$ and $\mathbf{P}$, the set of all equivalent objects in $\mathbf{E}$ at least one of whose attributes is possessed by T. In Pawlak's original work on rough sets the envelope and core were named 'upper approximation' and 'lower approximation' respectively. In the example if T = {blue egg}, then T^c = {egg}, T^e = {egg, blue cube, blue ball}. The pair $[T^c(\mathbf{P}, \mathbf{E}), T^e(\mathbf{P}, \mathbf{E})]$ is a *rough set*. The boundary of T is the set of equivalent objects in $\mathbf{E}$ in its envelope that are not in its core:

$$T^b(\mathbf{P}, \mathbf{E}) = T^e(\mathbf{P}, \mathbf{E}) - T^c(\mathbf{P}, \mathbf{E}) \quad (4)$$

and the indifference set is the set of all equivalent objects in $\mathbf{E}$ which are not involved in the description of T.

$$T^i(\mathbf{P}, \mathbf{E}) = \mathbf{P} - T^e(\mathbf{P}, \mathbf{E}) \quad (5)$$

Let the set of all rough sets that may be defined using $\mathbf{E}$ partitioned as $\mathbf{P}$ be denoted by $\mathbf{R}$. Consider $R = [R^c, R^e], R' = [R'^c, R'^e] \in \mathbf{R}$. It is simple to show that the following set theoretic relations hold where the symbol '~' stands for complement:

$$\begin{array}{llllll} (R \cup R')^c & \supseteq & R^c \cup R'^c & (R \cup R')^e & = & R^e \cup R'^e \\ (R \cap R')^c & = & R^c \cap R'^c & (R \cap R')^e & \subseteq & R^e \cap R'^e \\ (\sim R)^c & = & \sim(R^e) & (\sim R)^e & = & \sim(R^c) \end{array} \quad (6)$$

Note that $\sim R^c$ is $\sim(R^c)$ is shorthand for the complement of the core of R, and should be distinguished from $(\sim R)^c$, the core of the complement of R.

As Skowron and Grzymala-Busse (1991) and Wong *et al* . (1992) have pointed out, there is a close relationship between the theory of rough sets and the theory of evidence (Shafer 1976). Since detailed comparisons are available in the aforementioned papers, an exhaustive discussion of the similarities will not be given here. Instead some of the more obvious likenesses between the formalisms will be pointed out. Evidence theory is based upon the idea of a set of propositions, called the frame of discernment, which is usually written as Θ. This is equivalent to $\mathbf{A}$. A function $m: 2^\Theta \rightarrow [0, 1]$ takes as its domain the power set of Θ, associating numbers between 0 and 1 to subsets of Θ. These subsets may overlap. Now m cannot distinguish between the members of these subsets, so that two propositions are equivalent according to m, $\approx_m$, if they are members of the same subset. Thus m creates a partition on Θ, though this partition is more general than $\mathbf{P}$ since members of Θ may appear in more than one P_i.

2.2 Rough sets and uncertainty

The traditional view of uncertainty handling centres around the notion of subjective truth. A fact is uncertain because it is not possible to unambiguously determine whether or not it is true; there is either some degree of subjective observer accuracy to be taken into account, or the fact is true in some cases and false in others. This view results in reasoning systems in which all facts are quantified with a truth value which expresses the degree to which they are felt to be certain. The facts may be related to one another, either by relations that are themselves quantified and which allow the truth of one fact to be established from the truth of others, or by relations whose truth may be established from the truth of the related facts. The aim of mathematical formalisms constructed to deal with uncertainty is to ensure the correct propagation of such truth values, and they usually say little about how the values are established. Indeed, the values invariably come from outside the theory, being the result of some expert assessment of the situation that a given reasoning system is designed to cope with.

It is clear, however, that the values are not external to the problem, but stem from the relationships between the facts with which the reasoning system deals. This is where the uncertainty resides. The relationships between facts are not deterministic because the relationships that are enforced upon the facts are simply not the real ones; the facts that are identified as being important do not sit in exact relation to one another. If symptom A is not uniquely caused by the occurrence of disease B, saying that $B \rightarrow A$ with a value of 0.7 does not capture the fact that B is usually accompanied by A, sometimes by C as well, and that A also occurs when D is present.

In contrast, rough sets model this situation perfectly. The underlying assumption in rough set theory is that there is a set of basic concepts in terms of which every interesting fact may be described. There is a many to one mapping between the set of concepts and the observable facts and between the set of concepts and the set of hypotheses between which the system discriminates. This means, in general, that there is no exact relationship between the facts and the hypotheses. Instead, a fact will encompass concepts that do not apply to the hypothesis to which it is related, or will fail to cover all of the concepts that the given hypothesis embodies.

This view of uncertainty also fits in well with evidence theory. In evidence theory two measures of certainty are employed; belief and plausibility. The lower measure, belief, of a particular object, is the sum of the probability of all the attributes that the object is known to possess. The upper measure, plausibility, is the sum of the probability of all the attributes that the object is not known not to possess. The mapping of probability to sets of attributes reflects the lack of knowledge about exactly which attributes belong to which objects. The core of an object is the set of attributes whose probability would be summed to get the belief of the object, and the envelope is the set whose summed probability would be the plausibility. However, in the usual application of evidence theory, the attributes are propositional hypotheses, whereas in the use of rough sets outlined here, the objects described by the attributes are the hypotheses.

2.3 Rough sets and knowledge representation

The essential idea of using rough sets for knowledge representation is that the basic item, the object whose core and envelope are manipulated, is a item of knowledge rather than the object of Pawlak's original work. Thus the use of rough sets in knowledge representation will be considered from the stand-point of a knowledge base used to reason about a physical system. A knowledge item is the basic unit from which a knowledge base is constructed. In this case knowledge items will be considered to be logical propositions that may be associated with truth values, but in general they may be any atomic unit of any knowledge representation scheme.

Consider a set of knowledge items $\mathbf{E} = \{E_1, \ldots, E_m\}$ concerning the system being represented. For instance these might be the symptoms for which an explanation is sought, say {headaches, fever, spots, rash}. Along with these knowledge items there is a set of hypotheses $\mathbf{T} = \{T_1, \ldots, T_p\}$ such as {measles, tuberculosis, whooping cough} which may be related to the facts $\mathbf{E}$ by observation. For instance it may be decided that

$T_k \rightarrow E_i \wedge E_j$, a possible medical gloss being "measles implies fever and spots". Now, rather than quantify the underlying uncertainty in this rule by attaching a number to it, consider the set of underlying concepts $\mathbf{A} = \{A_1, \ldots, A_n\}$. Often these are unknown, and they rarely map one to one onto the E_j and T_k, the facts that the domain in question forces one to deal with. All that is known is that the A_i map onto the E_j so that one or more A_i relate to each E_j. Similarly, one or more concept relates to each T_k.

Because the concepts A_i are the fundamental concepts of the domain, there is an exact relationship between sets of A_i for every piece of information about that domain. For example, the rule relating to the symptoms of measles may be $A_4 \wedge A_5 \rightarrow A_1 \wedge A_2 \wedge A_3$. However, because the things that are observed do not map cleanly onto these concepts, and statements can only be made in terms of the things that are observable, it is only possible to make statements that are approximately true. Thus it is said that $T_k \rightarrow E_i \wedge E_j$ which for $E_i = A_1, E_j = A_3 \wedge A_6$ and $T_k = A_4$ amounts to $A_4 \rightarrow A_1 \wedge A_3 \wedge A_6$ which is clearly not quite right. Thus the use of observable facts to describe systems means that in general it is only possible to approximate the underlying concepts. Rough sets make it possible to keep track of these approximations since they relate the E_j and T_k to the A_i. If the E_j and T_k are given in terms of the A_i it is possible to write down logical expressions relating them, and manipulate them using the techniques described in Section 3. Alternatively it is possible to write down the logical relations between the E_j and T_k, based on an unknown set $\mathbf{A}$, from expert knowledge, quantifying the relations in much the same way as other certainty values (Saffiotti 1987) are obtained. In this case the propagation of the quantifiers is based on rough set theory, and thus takes into account the mismatch between the observable facts and the underlying concepts. This is covered in Section 4. Since all ideas of accuracy are relative to the set of basic concepts $\mathbf{A}$, it is possible to relate quantifiers based on rough sets to the kind of truth values discussed in other formalisms by careful choice of $\mathbf{A}$. This matter is discussed in Section 5.

3.0 Reasoning with rough sets

Having introduced the basic concepts of rough set theory, this section shows how rough sets may be used to define a new method for dealing with imprecisely known information.

3.1 Combining rough sets

The degree to which an object T may be defined within $(\mathbf{A}, \mathbf{E})$ may be determined from the cardinality of T^{c} and T^{e}. Pawlak (1984) gives the following:

If	$T^{c} = T^{e}$	then A is precisely defined by $(\mathbf{A}, \mathbf{E})$	
If	$T^{c} \neq T^{e}$ and $T^{e} \neq \emptyset$	then A is roughly defined by $(\mathbf{A}, \mathbf{E})$	
If	$T^{c} = \emptyset$	then A is internally undefined by $(\mathbf{A}, \mathbf{E})$	(7)
If	$T^{e} = \mathbf{E}$	then A is externally undefined by $(\mathbf{A}, \mathbf{E})$	
If	$T^{c} = \emptyset$ and $T^{e} = \mathbf{E}$	then A is totally undefined by $(\mathbf{A}, \mathbf{E})$	

It is possible to determine the degree to which logical combinations of roughly defined objects may themselves be defined. The usual logical operations of disjunction, $\vee$, conjunction , $\wedge$, and negation, $\neg$, may be defined in terms of set operations on the core and envelope of the objects concerned:

$$
\begin{array}{llllll}
(T\vee S)^c & = & (T\cup S)^c & (T\vee S)^e & = & (T\cup S)^e \\
(T\wedge S)^c & = & (T\cap S)^c & (T\wedge S)^e & = & (T\cap S)^e \\
(\neg T)^c & = & (\sim T)^c & (\neg T)^e & = & (\sim T)^e
\end{array} \qquad (8)
$$

where $\{\neg, \vee, \wedge\}$ have their usual meanings so that $\neg T$ means "not T", $T\wedge S$ means "both T and S", and $T\vee S$ means "either T or S or both". Using (6) we obtain:

$$
\begin{array}{llllll}
(T\vee S)^c & \supseteq & T^c\cup S^c & (T\vee S)^e & = & T^e\cup S^e \\
(T\wedge S)^c & = & T^c\cap S^c & (T\wedge S)^e & \subseteq & T^e\cap S^e \\
(\neg T)^c & = & \sim(T^e) & (\neg T)^e & = & \sim(T^c)
\end{array} \qquad (9)
$$

These results may be used to determine other logical combinations of roughly defined objects such as material implication, $\rightarrow$, where $T\rightarrow S \equiv \neg T\vee S$:

$$
(T\rightarrow S)^c \quad \supseteq \quad \sim T^e\cup S^c \qquad (T\rightarrow S)^e \quad = \quad \sim T^c\cup S^e \qquad (10)
$$

Note that the core of the disjunction of two terms may not be specified precisely since only the lower bound is ever known. Since the core is itself a lower bound on the accurate description, this is not a problem. Similarly, given the inequality in the description of the envelope of the conjunction, the envelope of any combination involving conjunction will only be defined by an upper bound. Again, this is not problematic.

Given the results of equations (8)–(10), it is possible to deduce the logical relationships between objects if the set of concepts $\mathbf{A}$ is known. For instance, consider a set of concepts $\mathbf{A} = \{A_1, A_2, A_3, A_4, A_5\}$, and a pair of objects E_1 and E_2 where $E_1 = [\{A_1, A_2\}, \{A_1, A_2, A_3\}]$, which is to say that the core of E_1 is $\{A_1, A_2\}$ and its envelope is $\{A_1, A_2, A_3\}$, and $E_2 = [\{A_2, A_3\}, \{A_2, A_3, A_4\}]$. In this case $T_1 = [\{A_2\}, \{A_2, A_3\}]$, is equivalent to $E_1\wedge E_2$, while $T_2 = [\{A_1, A_2, A_3\}, \{A_1, A_2, A_3, A_4\}]$, is equivalent to $E_1\vee E_2$. In a similar way it is possible to establish the validity of logical statements about objects whose rough descriptions are known. Given $E_3 = [\{A_3, A_4\}, \{A_2, A_3, A_4, A_5\}]$ and $E_4 = [\{A_2, A_3\}, \{A_1, A_2, A_3\}]$ it is clear that $E_3\rightarrow E_4$ will have the rough description $[\{A_1, A_2, A_3\}, \{A_1, A_2, A_3, A_5\}]$.

Similar definitions to those in (7) can be applied to evidence theory, remembering that T^c corresponds to the set of propositions over which belief in T is calculated, and T^e corresponds to the set determining the plausibility of T. If the core and envelope of a proposition are equal, then the belief in it is equal to its plausibility, and both measures collapse to a single precise probability. If the set over which the plausibility is calculated is non-zero, and the set over which the belief is calculated is non-zero and unequal to it, then the two measures provide bounds on the probability of the set. If belief is computed over the empty set then it is defined to be zero, and when plausibility is computed over Θ it is defined to be 1. When a hypothesis has belief 0 and plausibility 1 then nothing is known about it. The logical combinations of sets also transfer to evidence theory.

3.2 A logic for rough reasoning

The ideas introduced in the previous section can be adapted to create a quantified logic in which rough sets are used to model upper and lower bounds on the value of the propositions. The core and envelope are still composed of partitions of $\mathbf{E}$, the language with which objects may be described. Thus the quantifier of a proposition is determined by the accuracy with which it is determined by the set of partitions. The logic, named $\mathbb{R}L$, is propositional and is defined as follows (after Reeves and Clarke (1990)). $\mathbb{R}L$ includes the set of connective symbols $\{\neg, \rightarrow, \wedge, \vee\}$ introduced above, a set of punctuation symbols $\{(,)\}$ and a set of propositional variables $\mathbb{P}$. The set of sentences based on $\mathbb{P}$ forms the propositional language $L(\mathbb{P})$. Members of this set are defined by:

(1) Each of the elements of $\mathbb{P}$ is a sentence based on $\mathbb{P}$.
(2) If $\mathcal{S}$ and $\mathcal{T}$ are sentences based on $\mathbb{P}$, then so are $(\neg\mathcal{S})$, $(\mathcal{S}\rightarrow\mathcal{T})$, $(\mathcal{S}\wedge\mathcal{T})$ and $(\mathcal{S}\vee\mathcal{T})$.
(3) Nothing else is a sentence based on $\mathbb{P}$.

$\mathbb{R}L$ also has a set of axioms generated by the following axiom schemata. If $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ are any sentences, then:

(A1) $(\mathcal{S} \rightarrow (\mathcal{T} \rightarrow \mathcal{S}))$
(A2) $((\mathcal{S} \rightarrow (\mathcal{T} \rightarrow \mathcal{R})) \rightarrow ((\mathcal{S} \rightarrow \mathcal{T}) \rightarrow (\mathcal{S} \rightarrow \mathcal{R})))$
(A3) $(((\neg\mathcal{S}) \rightarrow (\neg\mathcal{T}) \rightarrow (\mathcal{T} \rightarrow S))$

There are three rules of deduction for $\mathbb{R}L$:

(MP) Modus ponens: from $\mathcal{S}$ and $\mathcal{S}\rightarrow\mathcal{T}$, for any $\mathcal{S}$ and $\mathcal{T}$, we obtain $\mathcal{T}$.
(MT) Modus tollens: from $\neg\mathcal{T}$ and $\mathcal{S}\rightarrow\mathcal{T}$, for any $\mathcal{S}$ and $\mathcal{T}$, we obtain $\neg\mathcal{S}$.
(R) Resolution: from $\mathcal{S}\vee\mathcal{T}$ and $\neg\mathcal{S}\vee\mathcal{R}$, for any $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$, we obtain $\mathcal{T}\vee\mathcal{R}$.

$\mathbb{R}L$ is quantified by defining a rough measure R on the propositional language of $\mathbb{R}L$, $L(\mathbb{P})$, such that $\forall p \in L(\mathbb{P}), R(p) = [p^{\leq c}, p^{\geq e}]$ where $p^{\leq c}$ is the lower bound on p^c, and $p^{\geq e}$ is the upper bound on p^e. $R(p)$ is thus an estimate of the degree to which p is defined by the set of partitions of **E**. The rough measure of formulæ may be established from the rough measures of the constituent sentences:

$$R(p\vee q) = [(p^c \cup q^c), (p^e \cup q^e)] \qquad R(p\wedge q) = [(p^c \cap q^c), (p^e \cap q^e)]$$
$$R(\neg p) = [(\sim p^e), (\sim p^c)] \qquad R(p\rightarrow q) = [(\sim p^e \cup q^c), (\sim p^c \cup q^e)] \tag{11}$$

Given these results, it is possible to determine how the rough measure is propagated when the rules of deduction of $\mathbb{R}L$ are applied. Firstly for modus ponens:

$$\begin{array}{lcl} R(p\rightarrow q) & = & [\alpha, \beta] \\ R(p) & = & [\gamma, \delta] \\ \hline R(q) & = & [\alpha\cap\gamma, \beta] \end{array} \tag{12}$$

Proof: $R(q) \geq R(p\cap q) = R((\neg p\vee q)\cap p) = [\alpha\cap\gamma, \beta\cap\delta]$ where the inequality $\geq$ is defined so that for $p, q \in P$ $R(p) \geq R(q)$ iff $p^c \subseteq q^c$ and $p^e \subseteq q^e$, so the lower bound on the core is $\alpha\cap\gamma$. In addition, $[\alpha, \beta] = R(\neg p\vee q) \geq R(q)$, so the upper bound on the envelope is β. Hence $R(q) = [\alpha\cap\gamma, \beta]$.■ A similar line of reasoning gives the pattern for modus tollens:

$$\begin{array}{lcl} R(p\rightarrow q) & = & [\alpha, \beta] \\ R(\neg q) & = & [\gamma, \delta] \\ \hline R(\neg p) & = & [\alpha\cap\gamma, \beta] \end{array} \tag{13}$$

and the way in which the propagation of rough measures occurs when the resolution rule of inference is used may also be determined:

$$\begin{array}{lcl} R(p \vee q) & = & [\alpha, \beta] \\ R(\neg p \vee r) & = & [\gamma, \delta] \\ \hline R(q \vee r) & = & [\alpha \cap \gamma, \beta \cup \delta] \end{array} \tag{14}$$

Proof: $R(q \vee r) = R((p \vee q \vee r) \wedge (\neg p \vee q \vee r)) = [((p \vee q \vee r)^c \cap (\neg p \vee q \vee r)^c), (p \vee q \vee r)^e \cap (\neg p \vee q \vee r))^e]$. Now, $(p \vee q \vee r)^c \supseteq (p \vee q)^c \cup r^c = \alpha \cup r^c$, and $(\neg p \vee q \vee r)^c \supseteq (\neg p \vee r)^c \cup q^c = \gamma \cup q^c$, so the lower limit on their intersection is $\alpha \cap \gamma$. Similarly, $(p \vee q \vee r)^e = (p \vee q)^e \cup r^e = \beta \cup r^e$, and $(\neg p \vee q \vee r)^e = (\neg p \vee r)^e \cup q^e = \delta \cup q^e$. Now, the upper limits on r^e and q^e are δ and β, respectively, so that the maximum value of the envelope is $\beta \cup \delta$.■

Once again ideas can be transferred to evidence theory. A frame of discernment Θ and a function m may be defined such that $\approx_m$ defines enclosing sets T^c and T^e for every proposition p in Θ. These sets will then combine exactly as described above when the various propositions are logically combined. This is a rather different approach from that adopted by Saffiotti (1990) in his belief function logic since the sets over which the belief and plausibility measures are defined are manipulated rather than the measures themselves.

4. Rough truth values

The rough measure described in Section 3 maintains the core and envelope of each distinct sentence, allowing a precise estimation of the degree to which it is defined by the system. In this section coarsenings of this measure are investigated. Instead of attaching a core and envelope to each sentence, the core and envelope are used to define a rough truth value for each sentence which is then propagated in place of the core and envelope.

4.1 Symbolic rough truth values

The rough measure of a sentence $p \in L(P)$ is determined by the degree to which its core and envelope are defined by the set of descriptors $\mathbf{A}$. The following boundary cases may be distinguished for $\emptyset \subset X \subset \mathbf{A}$, and $\emptyset \subset Y \subset \mathbf{A}$. These correspond to the definitions of (7).

If	$R(p)$	=	$[\mathbf{A}, \mathbf{A}]$	then p is true	
If	$R(p)$	=	$[X, \mathbf{A}]$	then p is roughly true	
If	$R(p)$	=	$[\emptyset, \mathbf{A}]$	then p is of unknown value	(15)
If	$R(p)$	=	$[\emptyset, Y]$	then p is roughly false	
If	$R(p)$	=	$[\emptyset, \emptyset]$	then p is false	

Similar definitions could be proposed for a measure based upon evidence theory. The rough values form a lattice, ordered by set inclusion $\subseteq$, giving the following:

$$[\emptyset, \emptyset] \subseteq [\emptyset, X] \subseteq [\emptyset, \mathbf{A}] \subseteq [Y, \mathbf{A}] \subseteq [\mathbf{A}, \mathbf{A}] \tag{16}$$

false   roughly false   unknown   roughly true   true

This suggests the introduction of a rough truth measure RV over $L(P)$ which identifies which of these five ordered states the rough measure of each $p \in L(P)$ falls into. The advantages of such a measures are its extreme simplicity and robustness, a direct result of the simple conditions used to define the values, and the fact that the values are ordered. The latter allows the axioms of the rough truth measure to be simply stated:

$$\begin{aligned} RV(p \vee q) &= \max(RV(p), RV(q)) \\ RV(p \wedge q) &= \min(RV(p), RV(q)) \end{aligned} \quad (17)$$

These may be easily verified by considering the set operations on the rough measure for each proposition. Similar considerations will validate the negation operator:

$RV(p)$	true	roughly true	unknown	roughly false	false
$RV(\neg p)$	false	roughly false	unknown	roughly true	true

(18)

The quantified logic may be applied without consideration for the set of basic concepts that underpin the rough values. However, the sound mathematical basis on which the logic is built ensures that it is well behaved.

It is possible to identify a further set of truth values. These are the contradictory values $R(\rho) = [\mathbb{U}, \emptyset]$, $R(\sigma) = [\mathbb{U}, \mathbb{Y}]$ and $R(\tau) = [\mathbb{X}, \emptyset]$, and the indeterminate value $R(\iota) = [\mathbb{Y}, \mathbb{X}]$. The first three are contradictory since, from the definition of core and envelope as being lower and upper approximations, respectively, it is clear that a set of rough values for p that have $p^c \supset p^e$ are contradictory to the underlying rough set theory. However, these values need not be considered since they are not generated by operations on the rough truth values introduced above.

4.2 Rough inference rules

In order to use the symbolic truth values for practical reasoning purposes, it is first necessary to provide a set of rules for propagating inference. It is trivial to establish the rough value of a material implication from that for disjunction:

$$RV(p \rightarrow q) = \max(RV(\neg p), RV(q)) \quad (19)$$

This is, however, of limited practical use since it only allows the computation of the strength of an implication from the strengths of its antecedent and consequent. Rules giving the strength of the consequent (antecedent) based on the strength of the material implication and its antecedent (consequent) are of more use in automated reasoning. To this end it is possible to adapt the reasoning patterns of (12), (13) and (14) , giving:

$$\begin{array}{ccccc} & & RV(p \rightarrow q) & = & \alpha \\ & & RV(p) & = & \beta \\ \hline \alpha & \geq & RV(q) & \geq & \min(\alpha, \beta) \end{array} \quad (20)$$

Proof: The upper limit is obtained by realising that $RV(p \rightarrow q)$ is the maximum of $RV(p)$ and $RV(q)$. The lower limit stems from the fact that $\min(\alpha, \beta) = RV((p \rightarrow q) \wedge p) = RV(p \wedge q) \leq RV(q)$. ■ A similar line of reasoning gives us the pattern for modus tollens:

$$\begin{array}{ccc} RV(p \rightarrow q) & = & \alpha \\ RV(\neg q) & = & \beta \\ \hline \end{array} \quad (21)$$

$\alpha \quad \geq \quad RV(\neg p) \quad \geq \quad \min(\alpha, \beta)$

These patterns bear a resemblance to those obtained for necessity weighted clauses (Dubois and Prade 1987). A rule for the propagation of rough truth values when sentences are resolved together may also be established:

$$\begin{array}{ccccc} & & RV(p \vee q) & = & \alpha \\ & & RV(\neg p \vee r) & = & \beta \\ \hline \max(\alpha, \beta) & \geq & RV(q \vee r) & \geq & \min(\alpha, \beta) \end{array} \tag{22}$$

Proof: $RV(q \vee r) = RV((p \vee q \vee r) \wedge (\neg p \vee q \vee r)) = \min(RV(p \vee q \vee r), (\neg p \vee q \vee r))$. Now, $RV(p \vee q \vee r) \geq RV(p \vee q) = \alpha$, and $RV(\neg p \vee q \vee r) \geq RV(\neg p \vee q) = \beta$, giving the lower limit. For the upper limit, consider the fact that $RV(p \vee q) = \max(RV(p), RV(q))$, so that the maximum value of $RV(q)$ is α, and the maximum value of $RV(r)$ is β. Since $RV(q \vee r) = \max(RV(q), RV(r))$, $RV(q \vee r) \leq \max(\alpha, \beta)$.■

Although, as stated above, a set of values similar to rough truth values could be proposed based upon the sets from which belief and plausibility are established in evidence theory, it appears that no such work has been carried out. The only measures based upon evidence theory that have been applied in the context of logic (Saffiotti 1990) (McLeish 1989) are numerical and contrast with the symbolic approach described here. The fact that this approach is symbolic stems from the decision to only deal with landmark values of the rough sets and thus the method shares some similarities with order of magnitude (Dubois and Prade 1989) and qualitative (Parsons 1993a) approaches. The combination rules are similar to those of possibilistic logic (Dubois, Lang and Prade 1987, 1989).

4.3 Numerical truth values

Pawlak *et al.*(1988) define a measure of the degree of dependency of $\mathbb{S}$ on $\mathbb{T}$ by the ratio of the core of $\mathbb{S}$ described by the concept set $\mathbb{T}$ to the full set of concepts. This measure is adopted here as the degree to which a fact is certainly true, and the notion is extended with a dual measure based upon the ratio of the envelope to the full set $\mathbf{A}$, which may be seen as a measure of the degree to which a fact may be true. The validity of these measures can be seen to follow from the possible worlds-based semantics for rough certainty values given in Section 5. A rough numerical measure RN is defined over the sentences $\mathcal{L}(\mathcal{P})$ of the logic $\mathbb{RL}$ such that $\forall p \in \mathcal{L}(\mathcal{P})$, $RN(p) = [p^l, p^u]$ where p^l and p^u are determined by the cardinality of p^c and p^e respectively:

$$\begin{array}{ccccccc} 0 & \leq & p^l & = & \frac{|p^c|}{|\mathbf{A}|} & \leq & 1 \\ 0 & \leq & p^u & = & \frac{|p^e|}{|\mathbf{A}|} & \leq & 1 \end{array} \tag{23}$$

The value of formulæ may be established from the measures of the constituent propositions. The following equations, which may be easily verified by considering the cardinalities of the sets in (9) and (10), are obeyed. For $RN(p) = [p^l, p^u]$, and $RN(q) = [q^l, q^u]$:

$$\begin{array}{lcl} RN(p \vee q) & = & [\max(p^l, q^l), \mathrm{Card}_{\cup}(p^u, q^u)] \\ RN(p \wedge q) & = & [\mathrm{Card}_{\cap}(p^l, q^l), \min(p^u, q^u)] \\ RN(\neg p) & = & [(1-p^u), (1-p^l)] \\ RN(p \rightarrow q) & = & [\max(1-p^u, q^l), \mathrm{Card}_{\cup}(1-p^l, q^u)] \end{array} \tag{24}$$

where $\text{Card}_{\cap}(x,y) = \begin{cases} 0 \text{ iff } x+y < 1 \\ x+y-1 \text{ otherwise} \end{cases}$ (25)

and $\text{Card}_{\cup}(x,y) = \begin{cases} 1 \text{ iff } x+y > 1 \\ x+y \text{ otherwise} \end{cases}$ (26)

The result for negation comes from $\neg p^{l} = \frac{|\sim p^{c}|}{|A|} = \frac{|A - p^{e}|}{|A|} = \frac{|A| - |p^{e}|}{|A|} = 1 - p^{u}$. A similar argument establishes $\neg p^{u}$. Given these results, it is possible to define the ways in which the numerical values are propagated across the inference rules of RL directly from the results of Section 3. It is possible to propagate rough numerical values across modus ponens:

$$\begin{array}{lcl} R(p \rightarrow q) & = & [\alpha, \beta] \\ R(p) & = & [\gamma, \delta] \\ \hline R(q) & = & [\text{Card}_{\cap}(\alpha, \gamma), \beta] \end{array} \quad (27)$$

modus tollens:

$$\begin{array}{lcl} R(p \rightarrow q) & = & [\alpha, \beta] \\ R(\neg q) & = & [\gamma, \delta] \\ \hline R(\neg p) & = & [\text{Card}_{\cap}(\alpha, \gamma), \beta] \end{array} \quad (28)$$

and the resolution rule:

$$\begin{array}{lcl} R(p \vee q) & = & [\alpha, \beta] \\ R(\neg p \vee r) & = & [\gamma, \delta] \\ \hline R(q \vee r) & = & [\text{Card}_{\cap}(\alpha, \gamma), \text{Card}_{\cup}(\beta, \delta)] \end{array} \quad (29)$$

The definition of the numerical truth values takes the approach back into line with evidence theory. As Skowron and Grzymala-Busse (1991) have shown, the lower and upper bounds of the measure RN are in fact belief and plausibility measures.

4.4 Reasoning using rough truth values

In the previous section, a number of rules for the propagation of rough truth values attached to the sentences of RL were established. These make it possible to use the quantified versions of RL to infer the truth of new facts from the truth of existing facts and relations. Thus a system using quantified RL can infer new information from its observations. This section examines the way in which the truth values of conclusions are related to the truth values of the facts on which the conclusions are based. The investigation is centred around the commonly used resolution rule, using the following adaptation of the approach taken by Dubois *et al*. (1989).

Given a set of clauses quantified with truth values derived from rough set theory, it is desirable to determine the bounds on the truth value of clauses derived by applying the resolution rule. Let C be a set of clauses of quantified RL, and $R(C)$ be the union of C with the results of resolving together every pair of clauses in C that may be resolved together. $R^{n}(C)$ denotes the result of iterating this procedure n times. It is clear that for

$C = \{C_1, ..., C_m\}$, where $\forall i = 1, ..., m$, $RV(C_i) \geq \alpha_i$, and C^n denotes any clause in $R^n(C)$, it is true that $\forall n \geq 0$, $RV(C^n) \geq \min_{i = 1, ..., m} \alpha_i$, and $\max_{i = 1, ..., m} \alpha_i \geq RV(C^n)$. Similarly, since any application of the resolution rule for the numerically quantified logic can result in a lower bound of 0 or an upper bound of 1, it is only possible to guarantee that $0 \leq RN(C_n) \leq 1$. Thus one may conclude that the truth value of the result of resolution between the members of a rough valued set of clauses must lie between the maximum and minimum values attached to members of that set of clauses, while the result of resolution between numerically quantified clauses is only constrained between 0 and 1.

While resolution may be used to reason 'forwards' from a set of known clauses in order to establish new facts, it is often used to reason 'backwards' in proofs by refutation. In proof by refutation, in order to assess whether a clause C_q follows from a set of clauses C the refutation rule is repeatedly applied to the set of clauses $C \cup \neg C_q$. If the empty clause $\{\}$ is obtained (i.e. by resolving a and $\neg a$) then the clause C_q follows from the set C. If this procedure is followed with RL quantified with RV and RN then there will be truth values attached to all clauses, including the empty clause at the end of the proof. It is desirable to establish the correspondence between the truth value of the empty clause and the truth value of the clause that is proved.

Let $C = \{C_1, ..., C_m\}$ be the set of clauses from which an attempt is being made to prove C_q. After the proof, $\beta \geq RV(\{\}) \geq \alpha$ (respectively $\beta \geq RN(\{\}) \geq \alpha$) is obtained from a subset of C, $C_\alpha \cup \{\neg C_q\}$, where $C_\alpha = \{C_i \mid \beta \geq RV(C_i) \geq \alpha\}$ ($C_\alpha = \{C_i \mid \forall j, C_j \in C_\alpha, \beta \geq Card_\cap(C_i, C_j), Card_\cup(C_i, C_j) \geq \alpha\}$) and $RV(\neg C_q) = t$ ($RN(\neg C_q) = 1$). Since $\{\}$ would be obtained from C_α by applying the classical resolution rule (ignoring degrees of truth), C_q is a logical consequence of C_α so that $RV(\neg(\cap_i \{C_i \in C_\alpha\}) \vee C_q) = t$ ($RN(\neg(\cap_i \{C_i \in C_\alpha\}) \vee C_q) = 1$). Now, $\beta \geq RV(\cap_i \{C_i \in C_\alpha\}) \geq \alpha$ ($\beta \geq RN(\cap_i \{C_i \in C_\alpha\}) \geq \alpha$), so applying the resolution rule to the two clauses gives us $t \geq RV(C_q) \geq \alpha$ ($1 \geq RV(C_q) \geq \alpha$). So the value of the empty clause is the lower limit of the value of the clause being proved.

4.5 Theorem proving using rough truth values

An example of reasoning using the rough valued resolution rule is the following adaptation of the 'meeting problem' presented by Dubois *et al.* (1987). The problem is restated here, the only change being that resulting from the limited number of values. Necessity values of 1 are replaced by rough truth values of true (t), necessity values of greater than 0.5 are replaced by rough truth values of roughly true (rt), and those values of less than 0.5 replaced by roughly false (rf). The following clauses, with the rough value of each clause in brackets, describe the problem:

C1. If Robert comes to the meeting tomorrow, then Mary will not come. $\neg R \vee \neg M$ (t).
C2. Robert is coming to the meeting tomorrow. R (t).
C3. If Beatrix comes to the meeting tomorrow, it is unlikely to be quiet. $\neg B \vee \neg quiet$ (rt).
C4. Beatrix may come to the meeting tomorrow. B (rf).
C5. If Albert comes to the meeting tomorrow, and Mary does not come, then it is almost certain that the meeting will not be quiet. $M \vee \neg A \vee \neg quiet$ (rt).
C6. It is likely that Mary or John will come tomorrow. $M \vee J$ (rt).

C7. If John comes tomorrow, it is rather likely that Albert will come. $\neg J \vee A$ (rt).

To ascertain if the meeting will be quiet, we add the clause:

C0. The meeting will be quiet. quiet (t).

This apparently cavalier treatment of values deserves some explanation. Firstly, there is no attempt to suggest an equivalence between the values. As Dubois and Prade (1987) point out, necessity measures are not degrees of truth, they are estimates of the degree of necessity that the proposition is true. The distinction between necessity values and rough truth values should thus be clear. All that is proposed is a suitable way of encoding the difference between the different certainties of the clauses. Secondly, the assignment of rt to $N(p) > 0.5$ and rf to $N(p) < 0.5$ is not an indication of ill-founded assumptions about the relative necessity values. Instead it indicates that, to maximise the limited discriminatory power of the rough valued logic, the lesser half of the possible necessity values will be modelled with the lower rough value, and the upper half with the higher.

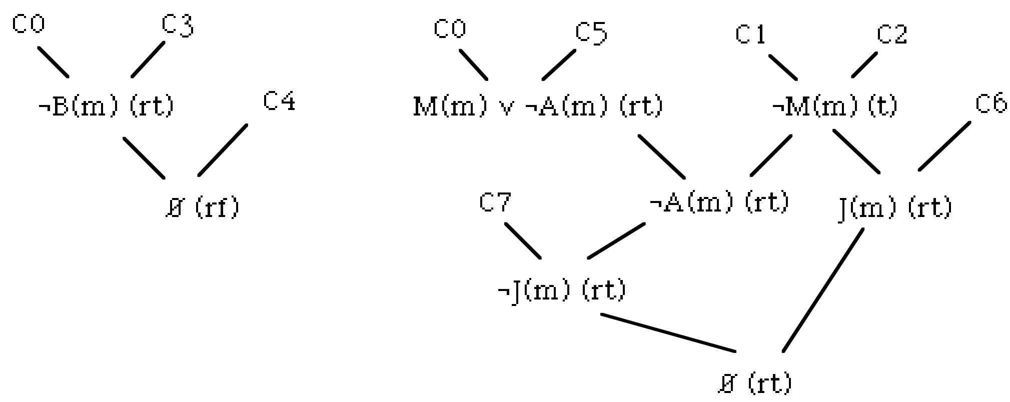

Figure 1. Two refutations for the meeting example.

There are two possible refutations of C0 (see Figure 1.). Of these, the second is the optimal, giving the necessary solution with a higher rough measure. The numerically quantified logic could be used to solve the problem, giving similar results to those obtained by Dubois *et al*.

5. A possible worlds semantics for rough truth values

This section introduces a possible worlds semantics for measures based on rough sets. This semantics is similar to that introduced by Carnap [1962] for probability theory, and subsequently adapted by Ruspini [1987] for evidence theory.

5.1 The Carnapian universe

This section is based upon Ruspini's [1987] description of Carnap's semantics. Carnap's approach involves the construction of a space of possible worlds that encompasses all valid states of a system of interest. First, all propositions of relevance to the system, a, b, c, d, ... are considered. All possible conjunctions of the type:

$$a \wedge \neg b \wedge \neg c \wedge d \wedge \ldots$$

where every proposition appears only once either as itself or as its negation, are then constructed. After discarding logical impossibilities, the resultant set of logical expressions includes all possible system states that may be represented using the propositions a, b, c, ... Each of these states corresponds to the truth of an atomic proposition about the system that is under consideration. Only one of the propositions may be true, and so the set of propositions describes all the possible different states of the universe. Each state thus represents a "possible world". If a possible world is viewed through what Ruspini calls a "conceptual microscope" so that the individual propositions may be discerned (Figure 2) it is clear that the possible world contains all true propositions in that world, including the negations of all of those that are false in that world. Two possible worlds will always be different since at least one proposition that is true in one will be false in the other.

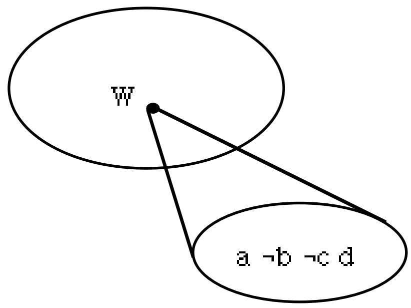

Figure 2. A close-up view of part of the Carnapian universe.

5.2 The universe for rough sets

To provide a semantics for rough set measures, a new set of possible worlds are constructed from the same set of propositions used by Carnap. Once again a set of conjunctions are built from the set of interesting propositions about a system, but this time it is possible for both a proposition and its negation to be missing from a conjunction. Thus, given the set of propositions a, b, c, the valid conjunctions include $a \wedge b$ and $\neg b$ as well as $a \wedge b \wedge \neg c$ allowing a much richer structure than is possible using Carnap's scheme. Once again logical impossibilities are discarded, giving a set of possible system states that are more extensive than those allowed by Carnap. Each of these states again corresponds to the truth of an atomic proposition about the system, and any two possible worlds will differ since at least one proposition which is true in one will either be false or missing in the other.

One of the possible worlds will correspond to the atomic proposition that is true, and this is chosen to be the possible world in which all of the propositions are present and true. One possible world will correspond to the contradiction, and this chosen to be the world in which all the propositions are present and false. This choice enables the set of attributes $\mathbf{A}$ to be taken as the set of interesting propositions a, b, c, d, ... A structure may be imposed upon the set of possible worlds by means of an relation $\mathcal{R}$ that specifies which possible worlds are accessible from a given possible world. $\mathcal{R}$ is chosen so that for every world w_i that is accessible from a given world w, the number of propositions in w and w_i differs by one. All those propositions in that are true in the world with fewer propositions are also true in the other, and all those propositions that are false in the world with fewer propositions are false in the other. The additional proposition in the larger world may be true or false. Thus the structure over the possible worlds is that of Figure 3.

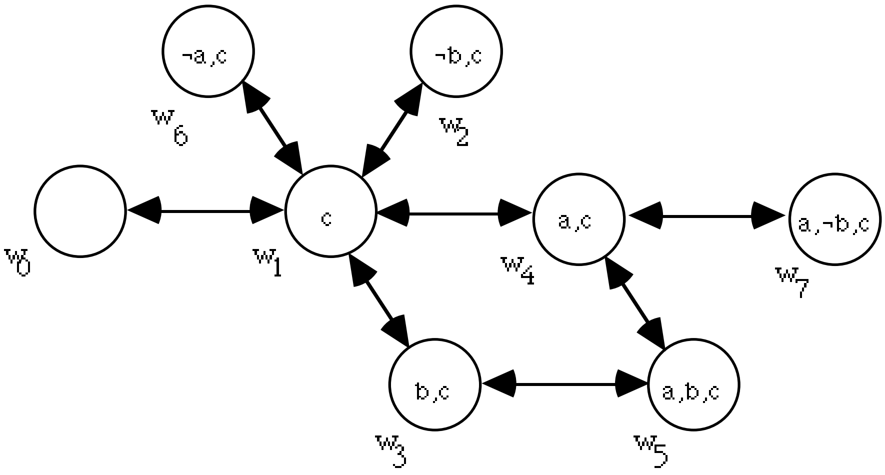

Figure 3. A structure over the set of possible worlds.

Figure 3 shows all the possible worlds accessible from w_1, but not all those accessible from the other worlds depicted. $\mathcal{R}$ is reflexive, transitive and symmetrical and thus corresponds to the accessibility relation used in modal logic S5 (Hughes and Cresswell 1968). The structure over the possible worlds is similar to that proposed by Moore (1985).

5.3 Rough truth values and possible worlds

Given $\mathcal{R}$, it is possible to establish the notion of the distance between two possible worlds. If a possible world is accessible from another possible world by several applications of $\mathcal{R}$, then the distance between them can be defined as the number of possible worlds through which the shortest path between them passes. Thus the distance between w_1 and w_2 in Figure 3 is zero, but between w_1 and w_7 is 1. The truth values attached to atomic propositions follow from the notion of distance. Given two possible worlds w_i and w_j, if there is a greater distance between w_i and w_T, the possible world that corresponds to the true atomic proposition, than between w_j and w_T, then the atomic proposition that corresponds to w_j has a higher truth value than the proposition that corresponds to w_i.

Now, the relative distance of two worlds from w_T can be determined from the number of true and false propositions contained in the world. The more true propositions contained by a world, the closer that world is to w_T. This may be seen in Figure 3 from the relative positions of w_1 and w_3 with respect to w_5 which is w_T for the worlds defined using the propositions a, b, and c. In a similar manner, the more false propositions that are contained in a world, the further from w_T it is. In Figure 3, w_2 is further from w_5 than w_1 since it has the additional false proposition $\neg b$. Since distance is related to truth it is possible to say that because w_3 has more true propositions in it than w_1, the atomic proposition corresponding to w_3 is closer to being true than that corresponding to w_1. It is also possible to say that since w_2 contains more false propositions that w_1, the atomic proposition corresponding to w_2 is further from the truth than that corresponding to w_1.

Having established that truth may be determined from the number of true and false propositions, it is possible to tie the idea of truth to the rough set notions of core and envelope. Consider an atomic proposition p. This is associated with a possible world w which contains a subset of a set of propositions $\mathbf{A}$. In particular w contains a conjunction of positive and negative attributes $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge \neg a_k \wedge \neg a_l \ldots$. It seems natural to make the core of p based upon $\mathbf{A}$, written p^c, the set of positive attributes that are contained in w, and the envelope of p based on $\mathbf{A}$, written p^e, the set of all members of $\mathbf{A}$ whose negation is not contained in w. Thus p^e contains all the members of $\mathbf{A}$ that are not contained in w either positively or negatively. The truth of p may then be established from p^c and p^e.

If w is w_T then p is the true proposition and $p^c = p^e = \mathbf{A}$. If w contains every proposition in $\mathbf{A}$ negated then p is the contradiction and $p^c = p^e = \emptyset$. For other w, p will be neither true nor false, and $\emptyset \subseteq p^c \subseteq p^e \subseteq \mathbf{A}$. For two propositions p and q with $p^c \subset q^c$ then p corresponds to a possible world that has fewer true propositions than the world that q corresponds. The world that p corresponds to is thus further from the real world than the world that q corresponds to, and so p is less true than q. Similar observations may be based on the knowledge that $p^e \subset q^e$. If $p^e \subset q^c$ then p is less true than q, but $p^c \subset q^e$ says nothing about the relative values. This may be stated more formally. For a pair of facts p and q, and any truth measure T based on the core and envelope values:

$$
\begin{array}{lllll}
T(p) & = & \text{true} & \text{if} & p^c = p^e = \mathbf{A} \\
T(p) & = & \text{false} & \text{if} & p^c = p^e = \emptyset \\
T(p) & \geq & T(q) & \text{if} & p^c \supseteq q^c \text{ and } p^e \supseteq q^e \\
T(p) & \leq & T(q) & \text{if} & p^c \subseteq q^c \text{ and } p^e \subseteq q^e \\
T(p) & = & T(q) & \text{if} & p^c = q^c \text{ and } p^e = q^e
\end{array}
\qquad (31)
$$

Clearly the extreme values of $T(p)$ are true and false, but there are a number of intermediate values. Indeed there is one such value for every possible world other than the real world and the possible world with no true propositions. The measures RV and RN considered in the previous section provide two possible ways of dividing up these intermediate values that seem particularly interesting. It should be noted that when $p^c \subseteq q^c$ and $p^e \supseteq q^e$ then it is possible for $T(p) > T(q)$, $T(p) = T(q)$ or even $T(p) < T(q)$. The exact relationship between $T(p)$ and $T(q)$ will be decided by the differences in size of the cores and envelopes, and the measure used. For instance if $\mathbf{A} = \{a, b, c, d\}$, $p^c = \emptyset$, $p^e = \{a, b, c, d\}$, $q^c = \{a\ b\}$ and $q^e = \{a\ b\ c\}$ then $RV(p) = u$ which is incommensurable with $RV(q)$ while $RN(p) = [0, 1]$ which is arguably less than $RN(q) = [0.5, 0.75]$ (see (Parsons 1993b) for a discussion of the relative magnitude of intervals).

6. Discussion

There are a number of points arising from the previous sections that are worthy of discussion. The first concerns the alternative view of uncertainty that this paper espouses, as the mismatch between what relates cause to effect and what is observed, that the rough set approach models. This seems to explain the presence of uncertainty in many domains quite neatly, but it remains to be seen if it is really helpful. Further work will help to resolve this concern. The second point concerns symbolic truth values which provide a simple method of handling uncertainty where only a few different truth values are required. These values have the interesting property of absorbing uncertainty in certain circumstances, in that the combination of a known value with the value "unknown" produces the known value. This is in contrast to other approaches to reasoning under uncertainty where combination with "unknown" values causes values to become less certain. This property is discussed in (Parsons *et al.* 1992). The third point concerns the numerical truth values introduced by the measure RN. The rule given for propagating these values in resolution has the unfortunate property of generating values which tend to [0, 1]. From this one might suppose that the numerical values are of little interest. However, since the values are belief and plausibility measures, it is possible that there are means of propagating RN in resolution, perhaps based on Dempster's rule, which mean that it does not tend towards [0, 1] so quickly. Further work will be directed towards discovering whether this is true. Finally, the consideration of a possible semantics for rough truth values indicates that there are a whole family of such sets of values. More work will be required to determine which, if any, of them are interesting from the point of view of reasoning under uncertainty.

7. Conclusion

This paper has presented several different ways in which rough sets may be used in reasoning under uncertainty. It is simple to define quantities in terms of rough sets, and formalisms based upon them are robust in the face of scarce information; a great advantage in dealing will ill-known domains such as biotechnology and ecology. Extending Pawlak's original work on rough sets this paper has given symbolic and numerical truth values based on rough measures, and supplied a semantics for them based on the concept of possible worlds. The symbolic values are simple and robust, being exceptionally good at assimilating ignorance, but suffer from being unable to precisely define the truth of the propositions with which they deal. The numerical values give precise truth values but are less robust, requiring accurate estimates of truth value. These values have been used to quantify a propositional logic, and rules for combining values when the inference rules of the logic are applied have been introduced. Some results have been given for theorem proving using the quantified logic. The use of rough sets has also been compared with the use of the theory of evidence and, despite the many similarities between the methods, rough sets alone make it possible to use a symbolic measure which is both more robust and simpler to understand than the numerical measures offered by evidence theory.

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