Adwords, An Algorithmic Perspective

Shoshana Neuburger

April 20, 2009
Grow your business on Google

No matter what your budget, you can display your ads on Google and our advertising network. Pay only if people click your ads.

Your ads appear beside related search results...

People click your ads...

...And connect to your business

Your ad here
See your ad on Google and our partner sites.

www.your-company-site.com
What are Adwords?

- Search engine displays search results.
What are Adwords?

- Search engine displays search results.
- For each query, relevant ads are also returned.
What are Adwords?

- Search engine displays search results.
- For each query, relevant ads are also returned.
- The search results and ads are displayed separately.
Some searches reveal many sponsored ads…
College and Dorm room Furniture and Rug Boston Massachusetts
Over 30 years College Furniture has supplied Boston Massachusetts colleges with dorm room furniture and the area with inexpensive furniture.
www.collegefurniturecheap.com/ - 11k - Cached - Similar pages -

College Furniture & Dorm Room Furniture - Free Shipping
College Furniture & Dorm Room Furniture The place to find great deals for dorm room furniture. We offer a wide selection and variety of college dorm room ...
www.onewayfurniture.com/college-furniture.html - 56k - Cached - Similar pages -

Dorm Essentials : College : Home : Target
Shop for Dorm Essentials College Home Products and Promotions at Target. ... Featured Items in College Furniture and Dorm Essentials.
College Furniture and ...
www.target.com/Furniture-Dorm-Essentials-College-Home/b?ie=UTF8&node=360124011 - 188k - Cached - Similar pages -

Dorm Furniture - College Bunk Beds and Computer Desks
Dorm Room Station offers sale priced college furniture, computer desks and bunk beds. Secure online shopping. Direct delivery.
www.dormroomstation.com/ - 52k - Cached - Similar pages -

School Furniture for Less
Shop our big selection of school furniture. Desks, chairs & more.
www.SchoolOutfitters.com

Student Travel Deals
We verify that you're a student. You Save.
www.StudentUniverse.com

Dorm Furniture
College Dorm Living Furniture Seating, Chairs, Futons, & Bedding!
www.Dormbuys.com

Buy Dorm Room Furniture
Studio Sofas Starting at just $325 Great Selection plus Free Shipping!
Foamiture.com

Dorm Furniture
Finest Solid Wood Furniture Built to Last, Quick Delivery!
www.JessCrate.com

College Furniture
Head to College in Style with Dorm
And others result in none...
4/20/2009
shoshana neuburger - Google Search

Web Images Maps News Video Gmail more ▼

Results 1 - 10 of about 1,140 for shoshana neuburger. (0.22 seconds)

FACULTY: Department of CIS: Brooklyn College CUNY
Shoshana Neuburger, shoshana@sci.brooklyn.cuny.edu. Konstantinos Nikolopoulos, costas@sci.brooklyn.cuny.edu. Arif Ozgelen, ozgelen@sci.brooklyn.cuny.edu ...
www.sci.brooklyn.cuny.edu/cis/main/faculty.html - 39k - Cached - Similar pages

Discrete Algorithms Seminar
occurrence of the pattern in the text thus far. Internet: http://portal.acm.org/citation.cfm?doid=1347082.1347201. Speaker: Shoshana Neuburger. ...
www.sci.brooklyn.cuny.edu/~amotz/802/f08-abstracts.pdf - Similar pages
More results from www.sci.brooklyn.cuny.edu »

Advanced Algorithms: Topics in Game Theory
Feb 16, MohammadTaghi, Market Clearing and Applications, Shoshana Neuburger. 4. Feb 23, Aaron, Inter-domain routing: Stable paths problem and dispute wheels ...
paul.rutgers.edu/~mangesh/cs514.html - 16k - Cached - Similar pages

Stringology 2009 - Bar Ilan University
... Moscow and LIFL, Lille); Laurent Mouchard, Université de Rouen; Joong Chae Na, Sejong University; Shoshana Neuburger, City University of New York ...
u.cs.biu.ac.il/~dombby/stringology/participants.php - 8k - Cached - Similar pages

Shoshana Neuburger Adwords, An Algorithmic Perspective
Advertiser

- An advertiser is charged for his ads.
An advertiser is charged for his ads.

Each advertiser can value each keyword differently.
An advertiser is charged for his ads.
Each advertiser can value each keyword differently.
An advertiser bids for a keyword that should display his ad.
Advertiser

- An advertiser is charged for his ads.
- Each advertiser can value each keyword differently.
- An advertiser bids for a keyword that should display his ad.
- Some keywords are more popular among advertisers.
Search engine’s perspective:

- Maximize revenue each day.
- Limitations:
  - must respect each advertiser’s daily budget and
  - the profit depends on an advertiser’s bid for a keyword he wins.

The bids and daily budgets are specified in advance.
Search engine’s perspective:

- Maximize revenue each day.
- Limitations:
  - must respect each advertiser’s daily budget and
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The bids and daily budgets are specified in advance.

Decision: which ads to display for a query?
The Adwords problem is an online allocation problem.
Online Algorithm

The Adwords problem is an **online** allocation problem.

The effectiveness of an online algorithm is measured by **competitive analysis**.
Online Algorithm

The Adwords problem is an online allocation problem.

The effectiveness of an online algorithm is measured by competitive analysis.

ALG is \( \alpha \)-competitive if the ratio between its performance and the optimal offline performance is bounded by \( \alpha \).

\[
\frac{\text{ALG}(I)}{\text{OPT}(I)} \geq \alpha \quad \text{for all instances I.}
\]
Greedy Criteria

Maximize the profit for each query.

As a keyword arrives, choose the ad that offers the highest bid.

Until the advertiser’s budget is depleted.
## Greedy Algorithm

### Example

<table>
<thead>
<tr>
<th></th>
<th>Bidder(_1)</th>
<th>Bidder(_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bicycle</td>
<td>$1$</td>
<td>$0.99$</td>
</tr>
<tr>
<td>Flowers</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>Budget</td>
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Greedy Algorithm

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Queries:
100 Bicycles then
100 Flowers.
## Greedy Algorithm

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Queries:
100 Bicycles
then
100 Flowers.

Greedy allocates 100 Bicycles to Bidder\(_1\).
Greedy Algorithm

Example

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</tr>
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Queries:
100 Bicycles
then
100 Flowers.

OPT (offline) allocates 100 Bicycles to Bidder_2 and 100 Flowers to Bidder_1.
Greedy Algorithm

**Example**

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Queries:

100 Bicycles
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<tr>
<th>Algorithm</th>
<th>Allocation</th>
<th>Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greedy</td>
<td>Bidder(_1): 100 Bicycles</td>
<td>$100</td>
</tr>
<tr>
<td>OPT</td>
<td>Bidder(_1): 100 Flowers</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bidder(_2): 100 Bicycles</td>
<td>$199</td>
</tr>
</tbody>
</table>
Competitive Analysis

ALG is $\alpha$-competitive if the ratio between its performance and the optimal offline performance is bounded by $\alpha$.

$$\frac{ALG(I)}{OPT(I)} \geq \alpha$$ for all instances $I$.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greedy</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Mehta, Saberi, Vazirani, Vazirani (’05)</td>
<td>$1 - \frac{1}{e} \approx .63$ optimal</td>
</tr>
</tbody>
</table>
AdWords and Generalized On-line Matching

Aranyak Mehta, Amin Saberi, Umesh Vazirani, Vijay Vazirani
FOCS, 2005
In search of a better algorithm

As a query arrives, the algorithm

- Should favor advertisers with high bids
- Should not exhaust the budget of any advertiser too quickly
In search of a better algorithm

As a query arrives, the algorithm

► Should favor advertisers with high bids
► Should not exhaust the budget of any advertiser too quickly

The algorithm weighs the remaining fraction of each advertiser’s budget against the amount of its bid.
Previous Results

Karp, Vazirani, Vazirani (1990)

- Online bipartite matching
- Randomized algorithm
  
  RANKING

- Fixes random permutation of bidders in advance.
- Budgets = 1, Bids = 0/1
- Factor: $1 - \frac{1}{e}$
Karp, Vazirani, Vazirani (1990)

- Online bipartite matching
- Randomized algorithm RANKING
- Fixes random permutation of bidders in advance.
- Budgets = 1, Bids = 0/1
- Factor: $1 - \frac{1}{e}$

Kalyanasundaram, Pruhs (2000)

- Online b-matching
- Deterministic algorithm BALANCE
- Matches query to bidder with highest remaining budget.
- Budgets = 1, Bids = 0/ε
- Factor: $1 - \frac{1}{e}$
"We saw online matching as a beautiful research problem with purely theoretical appeal. At the time, we had no idea it would turn out to have practical value."

- Umesh Vazirani, SIAM News, April 2005
Applying \textbf{BALANCE}

The new algorithm generalizes b-matching to arbitrary bids.

Natural Algorithm:
- Assign query to highest bidder
- Break ties with largest remaining budget

Achieves competitive ratio $< 1 - \frac{1}{e}$.
Applying BALANCE

The new algorithm generalizes b-matching to arbitrary bids.

Natural Algorithm:

- Assign query to highest bidder
- Break ties with largest remaining budget

Achieves competitive ratio \(< 1 - \frac{1}{e}\).

We would like to do better!
Adwords problem:

- $N$ bidders
- Each bidder $i$ has daily budget $b_i$
- Each bidder $i$ specifies a bid $c_{iq}$ for query word $q \in Q$
- A sequence of query words $q_1 q_2 \cdots q_M$, $q_j \in Q$, arrive online during the day.
- Each query $q_j$ must be assigned to some bidder $i$ as it arrives; the revenue is $c_{iq_j}$

Objective: maximize total daily revenue while respecting daily budget of bidders.
Simplified version of Adwords problem:

- Bidder pays as soon as ad is displayed
- Bidder pays his own bid
- One ad displayed per search page

Assumption: bids are small compared to budgets.
New Algorithm

Algorithm: Give query to bidder that maximizes

\[ \text{bid} \times \psi(\text{fraction of budget spent}) \]

\( \psi \) is tradeoff function between bid and unspent budget.

\[ \psi(x) = 1 - e^{-(1-x)} \]
Where does $\psi$ come from?

1 - $\frac{1}{e}$

New Proof for BALANCE $\{0, \varepsilon\}$

Factor Revealing LP

Modify LP for arbitrary bids $[0, \varepsilon]$

Use dual to get Tradeoff function

Tradeoff Revealing LP
Factor-Revealing LP

Choose large $k$.

Discretize the budget of each bidder into $k$ equal slabs.

Define the type of a bidder by the fraction of budget spent at end of BALANCE.

Define $x_1, x_2, \ldots, x_k$:

$x_i =$ number of bidders of type $i$. 
Factor-Revealing LP

\[ \text{w.l.o.g. } \text{OPT} = \$N. \]

\[ \text{Revenue} = \sum_{i=1}^{k} x_i \frac{i}{k} \]
Factor-Revealing LP

\[ \text{Revenue} = \sum_{i=1}^{k} x_i \frac{i}{k} \]

w.l.o.g. \( \text{OPT} = N \).

Constraint 1: \( x_1 \leq \frac{N}{k} \)

Constraint 2: \( x_1 + x_2 \leq 2\frac{N}{k} - \frac{x_1}{k} \)

\( \forall i, 1 \leq i \leq k - 1: \sum_{j=1}^{i} \left(1 + \frac{i - j}{k}\right)x_j \leq \frac{i}{k}N \)
Factor-Revealing LP

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{k} x_i \frac{i}{k} \\
\text{such that} & \quad \sum_{j=1}^{i} \left(1 + \frac{i - j}{k}\right)x_j \leq \frac{i}{k}N \\
& \quad \sum_{j} x_j = N
\end{align*}
\]
Factor-Revealing LP

Minimize \[ \sum_{i=1}^{k} \frac{x_i}{k} \]

such that \[ \sum_{j=1}^{i} \left(1 + \frac{i - j}{k}\right)x_j \leq \frac{i}{k}N \]
\[ \sum_{j} x_j = N \]

Solve LP by finding the optimum primal and dual.
Optimal solution is \( x_i = \frac{N}{k}(1 - \frac{1}{k})^{i-1} \)
which tends to \( N(1 - \frac{1}{e}) \) as \( k \to \infty \).
Thus, BALANCE achieves a factor of \( 1 - \frac{1}{e} \).
Where does $\psi$ come from?

1 - $\frac{1}{e}$

- New Proof for BALANCE \( \{0, \varepsilon\} \)
- Modify LP for arbitrary bids \([0, \varepsilon]\)
- Use dual to get Tradeoff function
- Use dual to get Tradeoff function
- Factor Revealing LP
- Tradeoff Revealing LP
Modify the LP for arbitrary bids

Subtle tradeoff between bid and unspent budget

We generalize LP $L$ and its dual $D$ to the case with arbitrary bids using LPs $L(\pi, \psi)$.

<table>
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<tr>
<th>$L(\pi, \psi)$</th>
<th>$D(\pi, \psi)$</th>
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<tbody>
<tr>
<td>$\max c \cdot x$</td>
<td>$\min b \cdot y$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$Ax \leq b + \Delta(\pi, \psi)$</td>
<td>$\min b \cdot y + \Delta(\pi, \psi) \cdot y$</td>
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Modify the LP for arbitrary bids

\[ \begin{align*}
L(\pi, \psi) & \quad \text{Max } c \cdot x \\
Ax & \leq b + \Delta(\pi, \psi)
\end{align*} \quad \begin{align*}
D(\pi, \psi) & \quad \text{Min } b \cdot y + \Delta(\pi, \psi) \cdot y \\
y^T A & \geq c
\end{align*} \]

Observation: For every \( \psi \), dual achieves optimal value at same vertex.
Modify the LP for arbitrary bids

\[
\begin{array}{c|c}
L(\pi, \psi) & D(\pi, \psi) \\
\hline
\text{Max } c \cdot x & \text{Min } b \cdot y + \Delta(\pi, \psi) \cdot y \\
Ax \leq b + \Delta(\pi, \psi) & y^T A \geq c \\
\end{array}
\]

Observation: For every \( \psi \), dual achieves optimal value at same vertex.

There is a way to choose \( \psi \) so that the objective function does not decrease.
Thus, the competitive factor remains \( 1 - \frac{1}{e} \).
Modify the LP for arbitrary bids

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Observation: For every $\psi$, dual achieves optimal value at same vertex.

There is a way to choose $\psi$ so that the objective function does not decrease.
Thus, the competitive factor remains $1 - \frac{1}{e}$.

This competitive factor is optimal.
More Realistic Models

The algorithm introduced by MSVV generalizes to

- Advertisers with different daily budgets.
- The optimal allocation does not exhaust all budgets.
- Several ads per search query.
- Cost-per-Click
- Second Price
Online budgeted matching in random input models with applications to Adwords

Gagan Goel and Aranyak Mehta
SODA, 2008
Outline of article

- Distributional assumption about query sequence: although the set of queries is arbitrary, the order of queries is random.
- Main Result: Greedy has competitive ratio $1 - \frac{1}{e}$ in the random permutation input model.
- This result applies to the i.i.d. model as well.
- Approach: modify KVV (fix hole in proof) and then apply results to Adwords problem.
Permutation Classes

For each item $p$, classify permutations

- Into those in which $p$ remains unmatched, *miss*.
- Into those in which $p$ gets matched.
  Subclasses depending on the structure of the match:
    - *good*: the correct match is available when $p$ arrives
    - *bad*: the correct match is allocated prior to the arrival of $p$
GM Algorithm

Properties of Greedy:
- Monotonicity
- Prefix
- Partition

These properties are simple observations in bipartite matching.

There can be several different bids for the same query in the Adwords problem.

Not every mismatch can be reversed easily; a tagging procedure is used to generalize the results.

The tagging method works on permutations of the input.
GM Algorithm

\[
\text{Revenue} = \sum_{i=1}^{m} \min \left\{ B_i, \sum_{q \in Q} \text{bid}_{iq} \right\}
\]

In the last bid the algorithm allocates to a bidder, his budget may be exceeded.

Inequalities bound the sizes of miss, bad, good.

A linear program maximizes the loss of revenue over these inequalities.

Factor revealing LP proves competitive ratio of \(1 - \frac{1}{e}\) in random input model.
GM Results

Competitive factor of Greedy in
- random-permutation input model
- independent distribution input model (i.i.d.)

is exactly $1 - \frac{1}{e}$. 
Thank you!