A proof theoretic tool for two first-order modal logics.[*]

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Abstract

One way to kickstart the proof theoretic study of a Hilbert-style modal logic has been to first "Gentzenise it", in the paradigm of [3, 4, 7], then prove that the associated Gentzen system admits cut-elimination, and finally use this tool to obtain metatheoretical results about the original logic.

We introduce here a "direct" syntactic tool that bypasses Gentzenisation and cut-elimination processes. The tool is a suite of well chosen *maps from modal formulae to modal formulae* —formula translators or "*formulators*"— that preserve provability.

We apply the tool to the proof theory of two modal predicate logics, one of which, M^3, appeared originally in [5, 6] and formalised some of the metatheory of first order classical predicate logic, allowing the modal box $\Box$ to simulate the informal $\vdash$. Precisely, we have $A \vdash B$ classically —from additional (to A) assumptions Γ— iff $\Box A \to \Box B$ can be proved modally from Γ and the set of all $\Box C$, where $C \in \Gamma$. This we shall call the *Conservation Result for* M^3.

More recently we introduced a closely related modal predicate logic as a common extension of M^3 and GL (of [1, 2, 3, 4, 7]). We will call it ML^3.

While [5, 6] proved the conservation result for M^3 *semantically* using Kripke structures, this talk will derive the result for *both* M^3 and ML^3 using the *syntactic* formulator technique. We will also prove that the *reflection principle* holds for both logics, namely, if $\Box A$ is provable, then so is A. Finally, we will also establish the negative results that neither *strong necessitation* ($A \to \Box A$) nor *strong reflection* ($\Box A \to A$) are provable in either M^3 or ML^3.

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Bibliography

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