On the Size and the Approximability of Minimum Temporally Connected Subgraphs

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Motivation

Network Properties are Time-Dependent

Graphs are used for modeling **networks** (e.g., transportation, communication, social) that are **dynamic** in nature.

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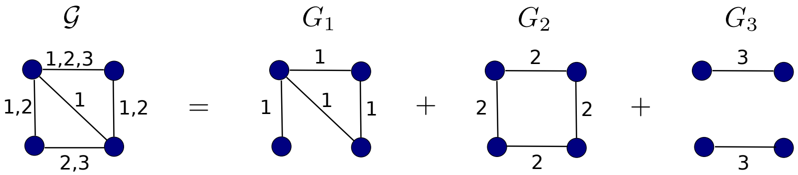
- Transportation and communication networks: congestion, maintenance, temporary failures.
- Social networks: relationships evolve with time.
- Networks modelling information spreading, epidemics, dynamical systems, ...

Temporal Graphs

- Generalized model that captures **network changes** over time.
- **Temporal Graph** : sequence $\mathcal{G} = (G_t(V, E_t))_{t \in [L]}$ of (undirected) graphs on vertex set V, edge set E_t varies with time t.
  - Edge e has set of (time)labels $l_1, \ldots, l_k$ denoting when e is **available** .



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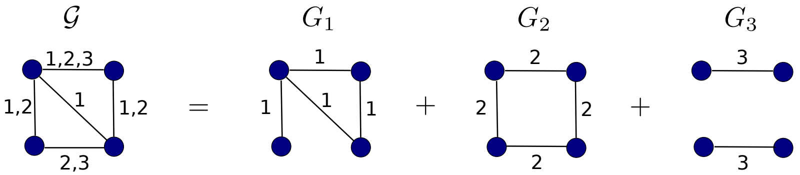
  - Maximum label L is the **lifetime** of $\mathcal{G}$.
  - Order $n = |V|$ and size $M = \sum_{t \in [L]} |E_t|$.



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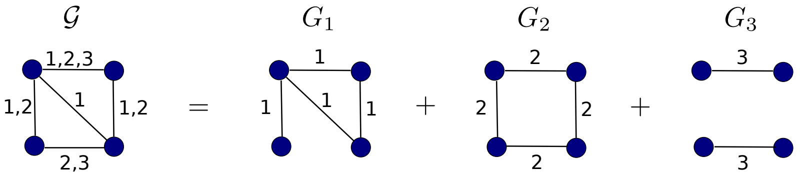

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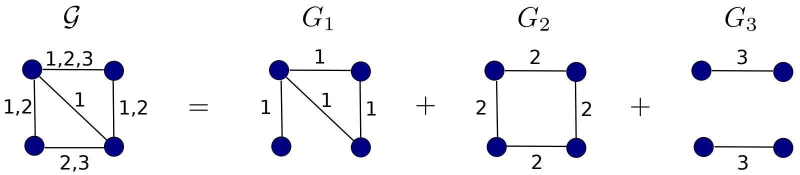
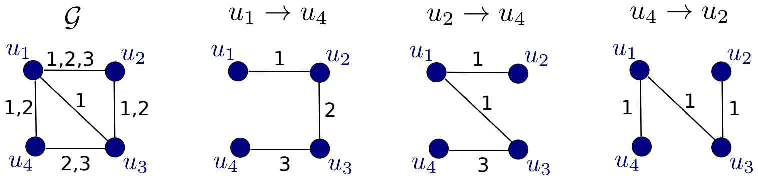
  - $\mathcal{G}$ can be edge (or vertex) **weighted**.
  - **Simple** if every edge available at most once.



Temporal $u_1 - u_k$ path: edge labels are **nondecreasing**.

- Temporal path $p = (u_1, (e_1, t_1), u_2, (e_2, t_2), \ldots, (e_{k-1}, t_{k-1}), u_k)$, where $t_i \le t_{i+1}$ and $e_i = \{u_i, u_{i+1}\} \in E_{t_i}$.



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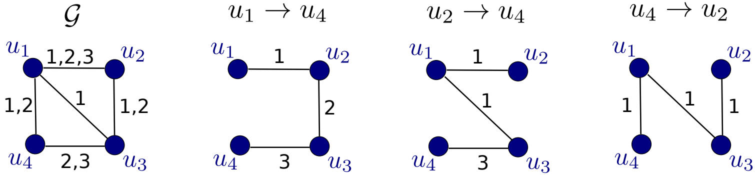
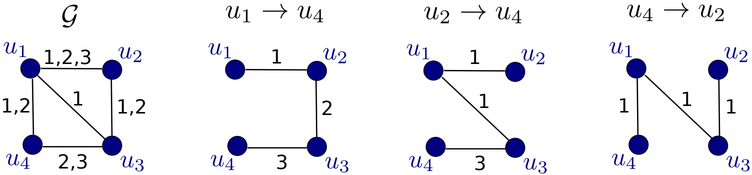
- Starting at u_1, we reach u_k by crossing edges only when **available** .
- We can **wait** at any vertex until an adjacent edge is available.
- Crossing an edge is **instant** .



Temporal Connectivity

- $\mathcal{G}$ is **s-temporally connected** , $s \in V$, if exists temporal $s - v$ for any vertex v.
- $\mathcal{G}$ is **temporally connected** if both $u - v$ and $v - u$ temporal paths exist for every vertex pair u, v.



Some Previous Work

- Model, temporal reachability, temporal version of **Menger's theorem** for edge (s,t)-connectivity [Berman 96]
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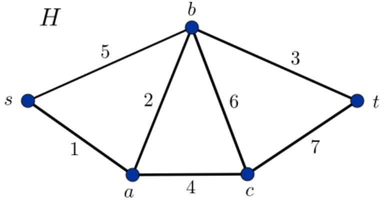
  - max # vertex disjoint $s-t$ paths = min # vertices whose removal separates s and t.



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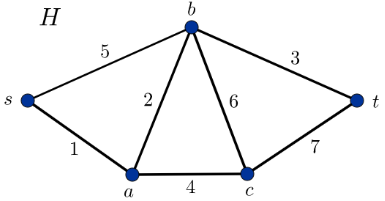

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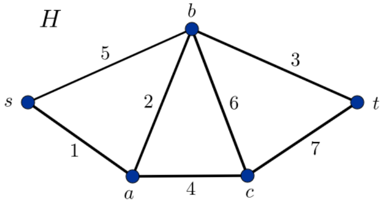
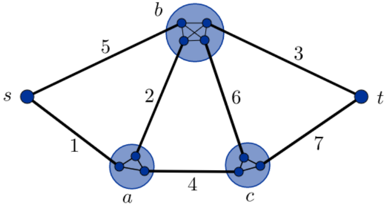
- Menger's theorem **holds** if **vertices** are also regarded as **temporal** [Mertzios Michail Chatzigiannakis Spirakis 13]





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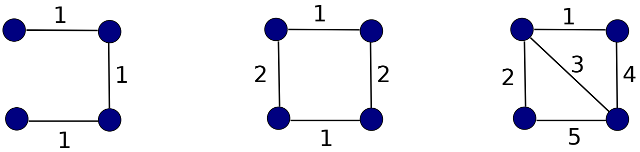

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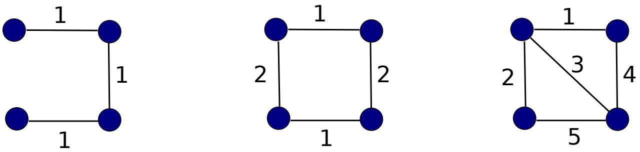

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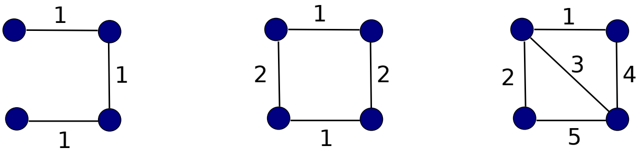
- **Temporal** graphs: temporal connectivity certificates **more complicated** and of **different size** .



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Upper and lower bounds on size of **temporal** connectivity certificates in worst case (for simple graphs)? [Kempe Kleinberg Kumar 00]

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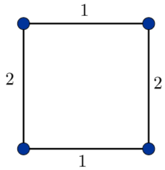
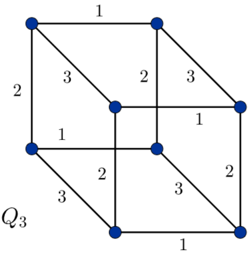
- Lower bound: temporal **hypercube** requires $\Omega(n \log n)$ edges.

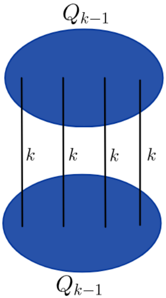


Q_2



Q_3

Q_k



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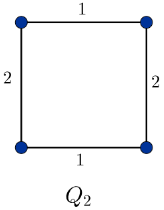
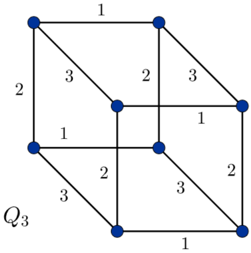
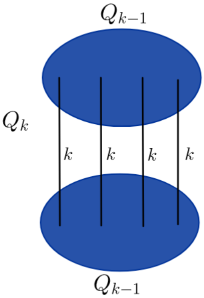
Upper and lower bounds on size of **temporal** connectivity certificates in worst case (for simple graphs)? [Kempe Kleinberg Kumar 00]

- (Trivial) upper bound: $O(n^2)$ (take n different v_i-rooted trees).
- Lower bound: temporal **hypercube** requires $\Omega(n \log n)$ edges.
- We **improve** lower bound to $\Omega(n^2)$!



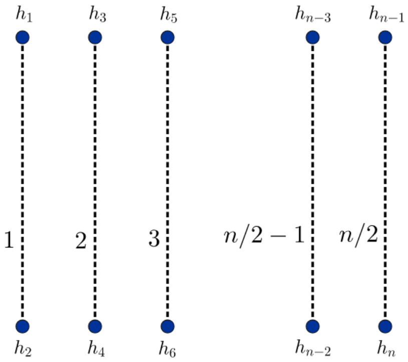




Quadratic Temporal Connectivity Certificates

Dense temporally connected graph where deletion of any edge breaks temporal connectivity.

- $n/2$ vertex pairs connected by $n/2$ **edge-disjoint** paths of **length** n each with a **different** label.



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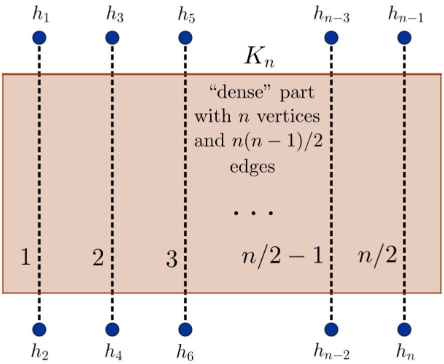
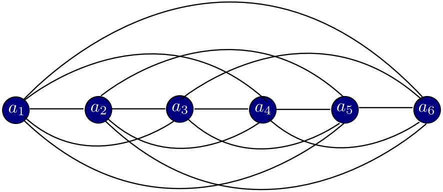
- Paths use the **same set** of n **intermediate** vertices.



Quadratic Temporal Connectivity Certificates

- Dense part: $n/2$ **edge-disjoint** paths of **length** n on same set of intermediate vertices.
- **Partition** a complete graph K_n into $n/2$ **Hamiltonian paths** .



Quadratic Temporal Connectivity Certificates

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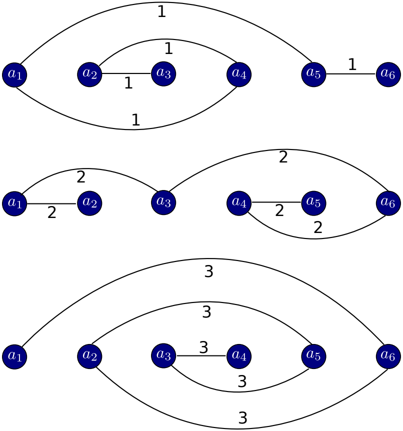
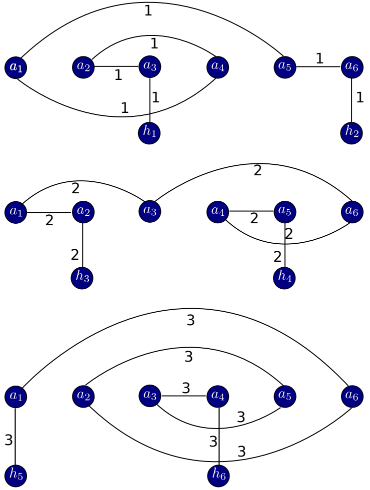


Quadratic Temporal Connectivity Certificates

- Attach 2 new vertices to the endpoints of each Hamilton path.
- All $n+1$ edges of the **i-th Hamilton path** have the **same label i** .



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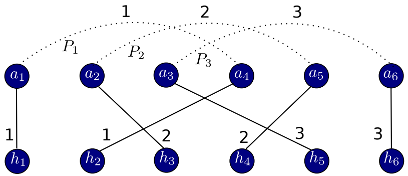
- Dense part: $n/2$ **edge-disjoint** paths of **length** $n+1$ on same set of intermediate vertices.
- Temporal paths $h_{2i} - h_{2i-1}$ and $h_{2i-1} - h_{2i}$ use edges with **label** i.



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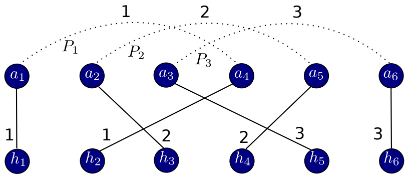
- Vertices $h_1, \ldots, h_{2i-2}$ **unreachable** from vertices h_{2i-1} and h_{2i}.



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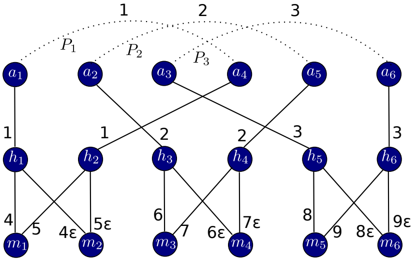

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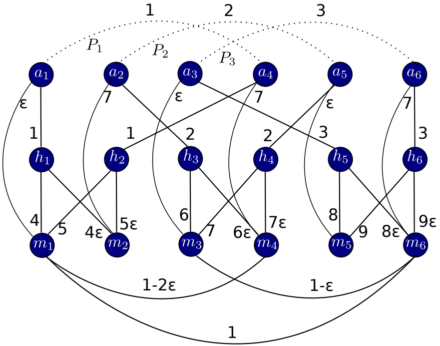
- Do **not** introduce **alternative** temporal $h_{2i} - h_{2i-1}$ paths (careful use of timelabels).
- m-vertices serve as **"entry"** and **"exit"** points of corresponding Hamilton path.



Quadratic Temporal Connectivity Certificates

- Temporal path $h_{2i} - h_{2i-1}$ uses edges with label i only.
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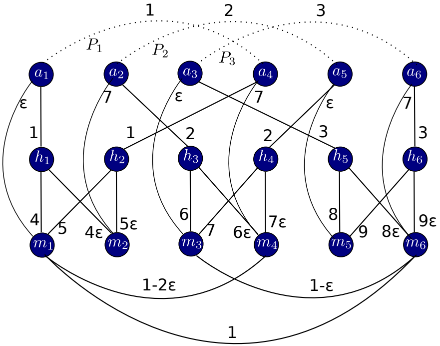

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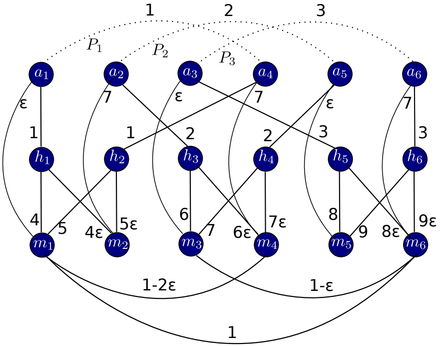
- **Linear** connectivity certificate by **changing a single label** !



Minimum Temporal Connectivity Certificate

Minimum Temporal Connectivity (MTC)

Given **connected edge-weighted** temporal graph $\mathcal{G}(V, E, w)$, find spanning subgraph $\mathcal{G}'(V, E', w)$, where $E'_t \subseteq E_t$ for all $t \in [L]$, of **minimum total weight** $\sum_{t=1}^{L} w(E'_t)$ and

- **Minimum s-Temporal Connectivity** (s-MTC) : $\mathcal{G}'$ is s-temporally connected.
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Both problems are **hard** to approximate:

- Temporal paths are inherently **directed** .
- Labels restrict relative order of edges in a path.
- Temporal connectivity similar to **Directed Steiner Tree / Forest** !

Minimum s-Temporal Connectivity (s-MTC)

Approximating s-MTC

- Optimal solution is a **tree**: $n - 1$ edges suffice.
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- Reduction to DST: approximation ratio $O(n^{\varepsilon})$, for any $\varepsilon > 0$, and $O(\log^3 n)$ in quasi-P [Charikar et al. 99]

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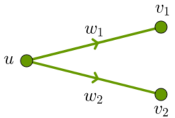
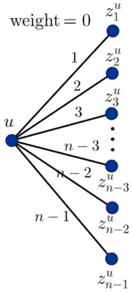
- Poly-time solvable if underlying graph has **bounded treewidth** .

Reduction from Directed Steiner Tree

Directed graph $H(V_H, E_H, w)$, $|V_H| = n$, source s, set of terminals T.

- Every vertex $u \in V_H$ becomes **vertex** u of temporal graph $\mathcal{G}$.
  - Temporal edges $(\{u, z_u^l\}, l)$ of weight 0, for $l = 1, \ldots, n-1$.





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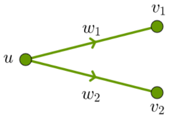

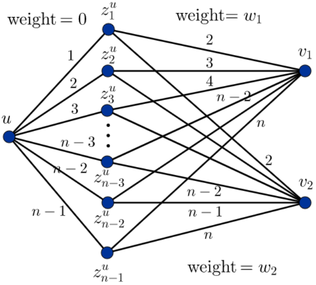

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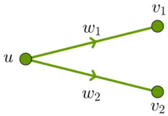
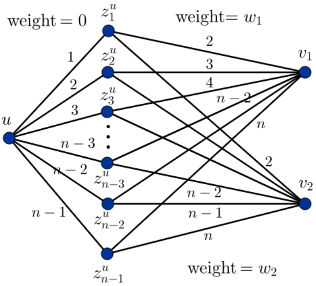
- s has **"direct" edge** with **label** $n+1$ and weight 0 to every z**-vertex** and to every **non-terminal** vertex .
  - $\mathcal{G}$ has $O(n^2)$ **vertices** and $O(n|E_H|)$ **temporal edges** .





Minimum Temporal Connectivity (MTC)

Approximating Minimum Temporal Connectivity

- Reduction from $(1, 2)$-Steiner tree: APX-hard for **unweighted** temporal graphs.
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- Reduction to Directed Steiner Forest : $O((\Delta M)^{2/3+\varepsilon})$-approximation [Feldman Kortsarz Nutov 12]

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- Complexity and approximability if **timelabels** are determined by **simple rules** ?

Thank You!