On the Quantitative Hardness of CVP

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Outline

- Closest Vector Problem
- Applications
- Hardness
- Isolating Parallelepipeds

The Closest Vector Problem

Lattice

- A lattice $\mathcal{L}$ is the set of all integer combinations of linearly independent basis vectors $\vec{b}_1, \ldots, \vec{b}_n \in \mathbb{R}^d$

$$\mathcal{L} = \mathcal{L}(\vec{b}_1, \ldots, \vec{b}_n) := \left\{ \sum_{i=1}^{n} z_i \vec{b}_i \; : \; z_i \in \mathbb{Z} \right\}$$

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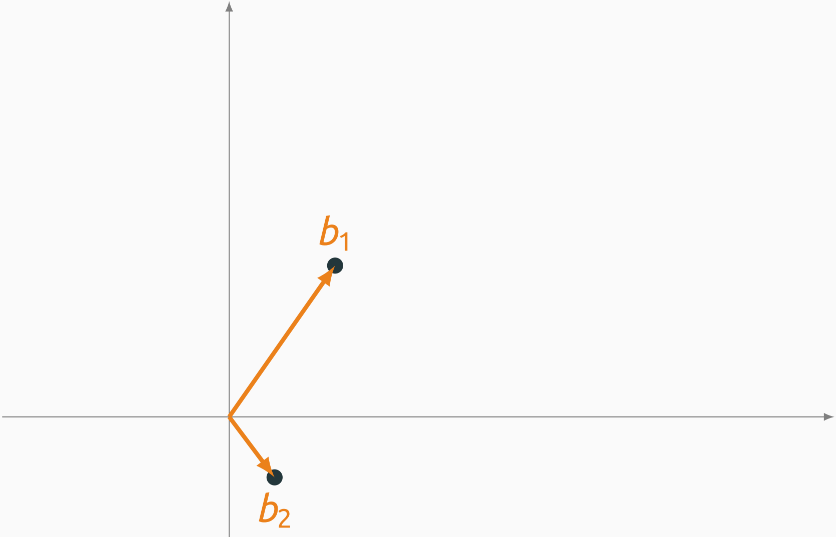
- n is the rank of $\mathcal{L}$, d is the (ambient) dimension

Lattice. Example



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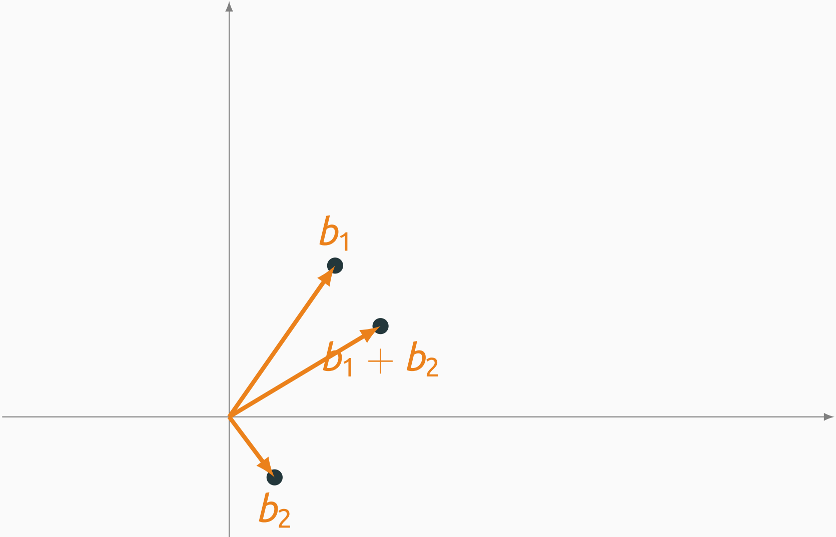

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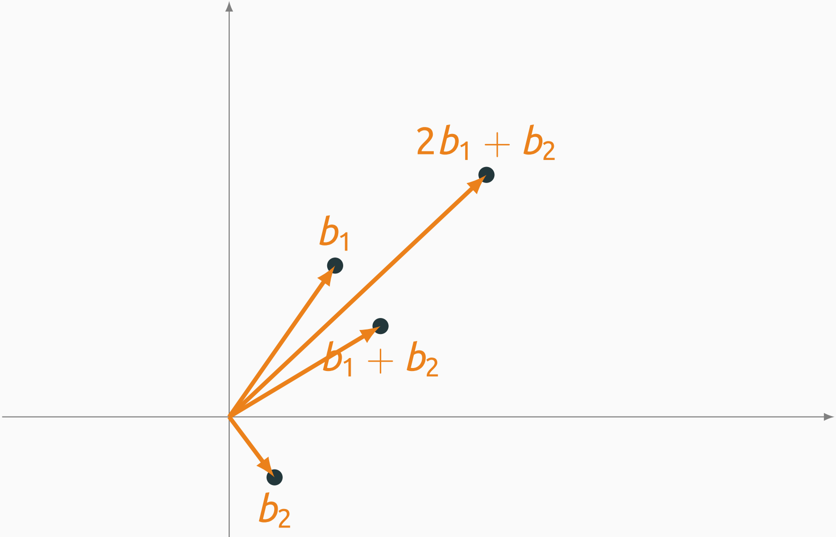

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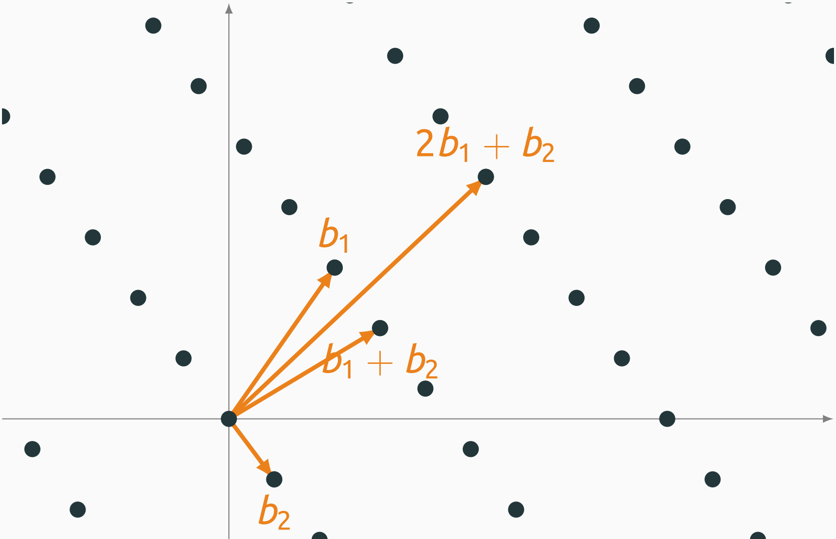

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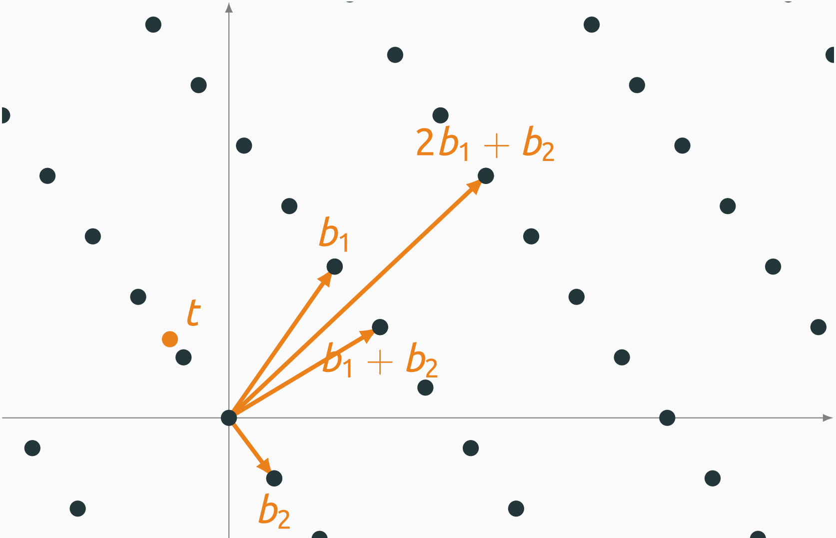

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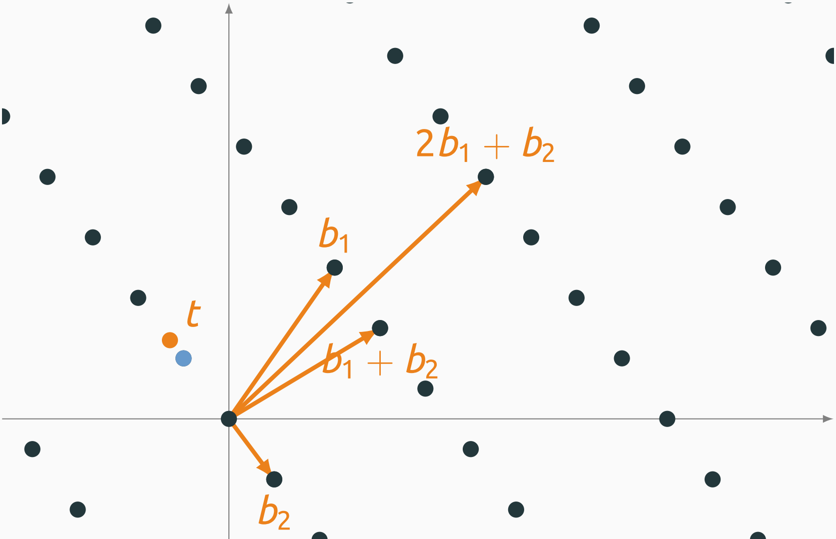

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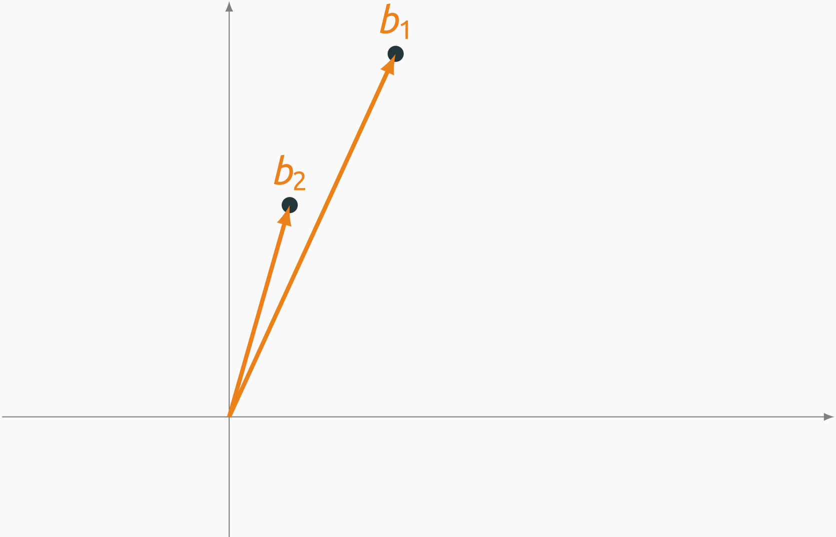

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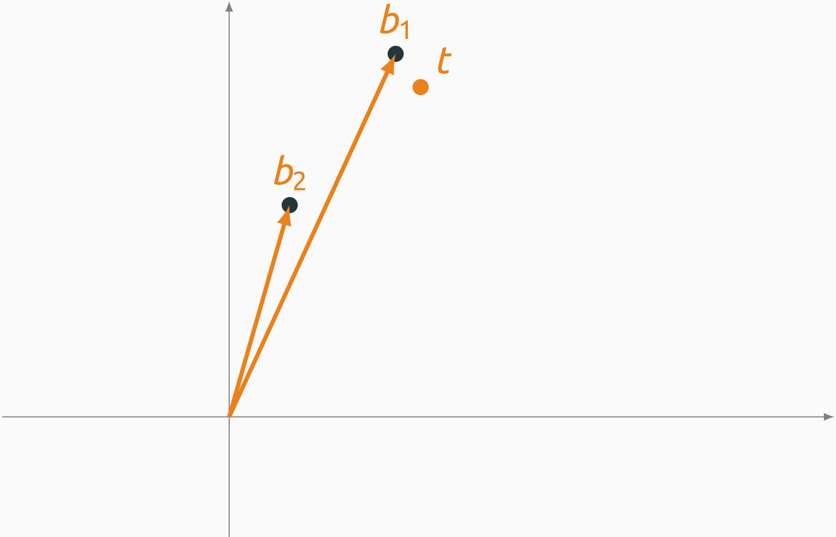

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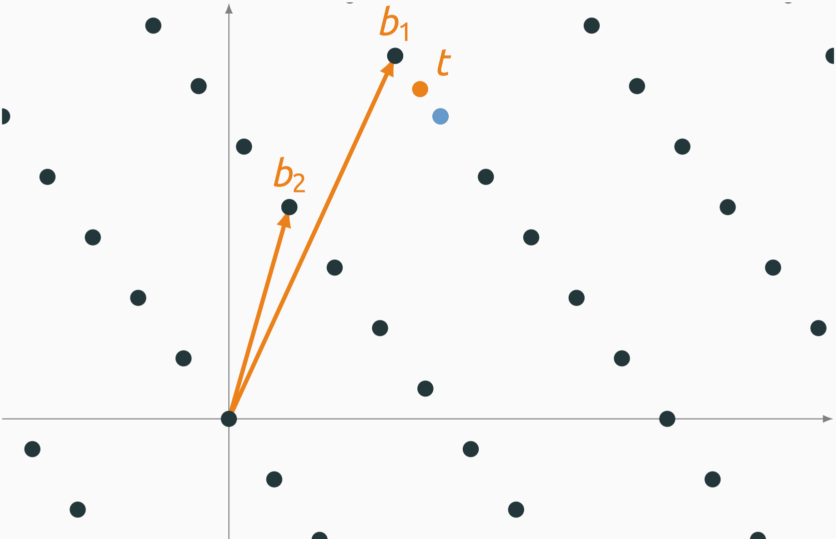

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$$\|\vec{x}\|_p := (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{1/p}$$

for $p = \infty$:

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- CVP_p —Closest Vector Problem in the ℓ_p norm

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- Cryptanalysis [Odl90,JS98,NS01]

Lattice-Based Cryptography

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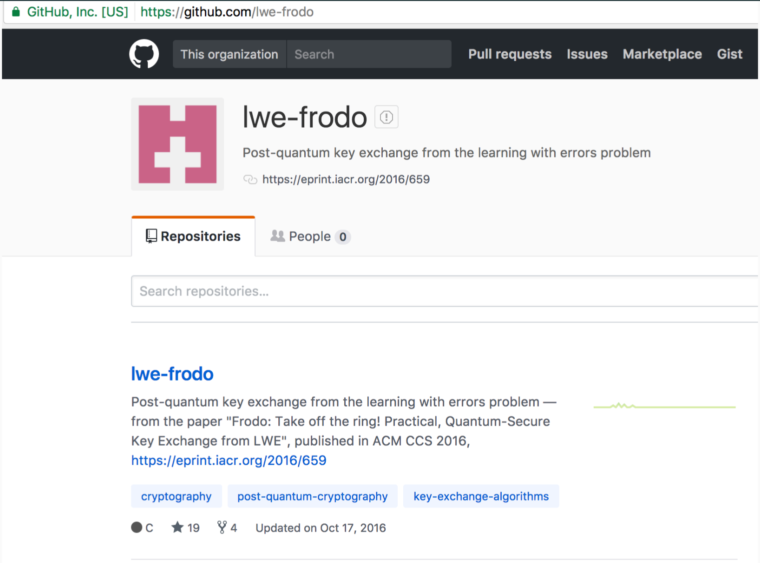
- About to be Deployed

Real Life Cryptography



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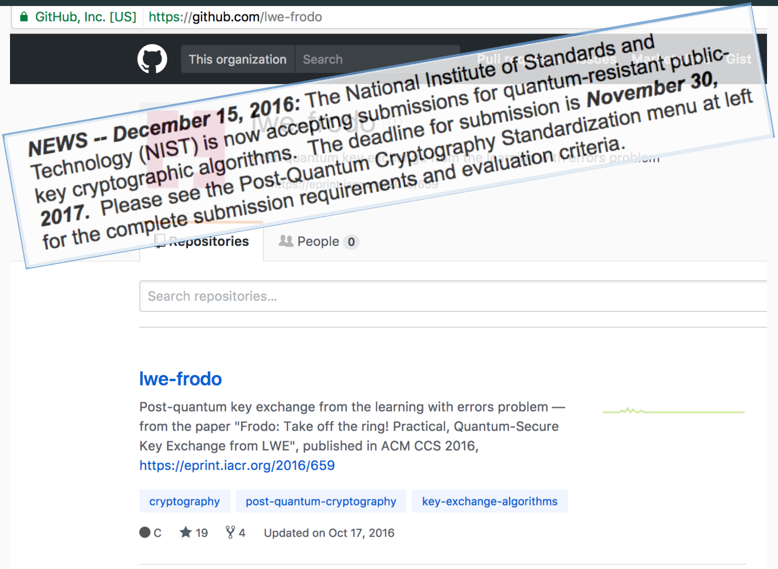

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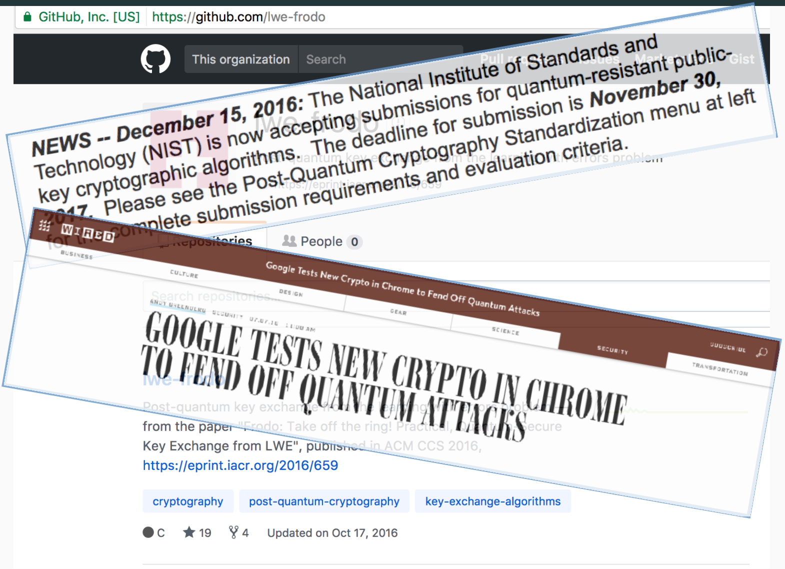

Hardness

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- CVP_2 can be solved in $2^{n+o(n)}$ time [ADS15]
- Cryptographic applications require quantitative hardness of CVP [ADPS16,BCD+16,NIS16]:
  a $2^{n/20}$-time algorithm would break these schemes in practice

k-SAT

- $(x_1 \lor \neg x_2 \lor \ldots \lor x_k) \land \ldots \land (x_7 \lor \neg x_4 \lor \ldots \lor x_3)$

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- SETH [IP99]. There exists a constant k:
  no algorithm solves k-SAT in $2^{0.99n}$ time
- Goal: Reduce k-SAT on n vars
  to CVP on a rank-n lattice

A Very Special Case: 2-SAT

$$
\begin{array}{c|cccccc}
 & x_1 & x_2 & \cdots & x_{n-1} & x_n & \\
\hline
x_1 & 2\alpha & 0 & \cdots & 0 & 0 & \alpha \\
x_2 & 0 & 2\alpha & \cdots & 0 & 0 & \alpha \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 & \vdots \\
x_n & 0 & 0 & \cdots & 0 & 2\alpha & \alpha \\
\hline
C_1 = (x_1 \vee x_2) & 2 & 2 & \cdots & 0 & 0 & 3 \\
C_2 = (x_1 \vee x_n) & 2 & 0 & \cdots & 0 & 2 & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
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	x_1	x_2	$\cdots$	x_{n-1}	x_n	
x_1	2α	0	$\cdots$	0	0	α
x_2	0	2α	$\cdots$	0	0	α
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	0	$\vdots$
x_n	0	0	$\cdots$	0	2α	α
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If $x \notin \{0,1\}^n$, distance is $\geq (n+1)\alpha^p$

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unsat clause contributes $3^p > 1$

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- Best algorithm for MAX-2-SAT runs in $2^{\omega n/3} < 1.74^n$

Generalization to k-SAT?

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- 2 and 4 are equidistant from 3!
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  - for all non-zero $\vec{y} \in \{0, 1\}^k$, $\|V\vec{y} - \vec{t}\|_p = 1$
  - for $\vec{y} = 0^k$, $\|V\vec{y} - \vec{t}\|_p = \|\vec{t}\|_p > 1$

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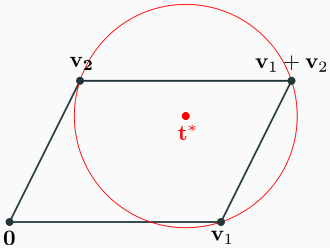

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Isolating Parallelepipeds in ℓ_1

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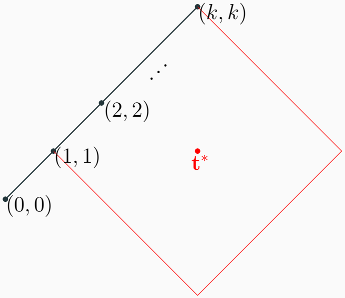
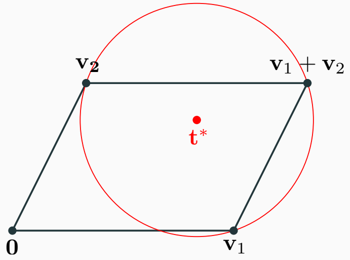


Isolating Parallelepipeds in ℓ_2



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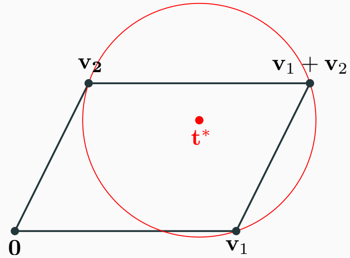
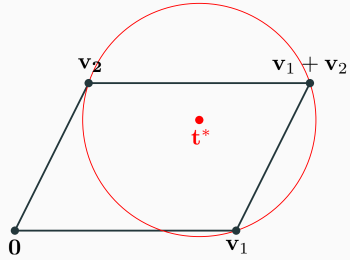


Can we do for 3 vectors?

Isolating Parallelepipeds in ℓ_2



Can we do for 3 vectors? No!

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- If p is an even integer, then IPs exist only for at most $k \leq p$ vectors
- For any k and any $p = p_0 + \delta(n)$ with $\delta(n) \neq 0$ and $\delta(n) \to 0$, they exist for sufficiently large n
- For any fixed k, IPs exist for all but finitely many values of p

Candidate for odd p

$$V := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \qquad \vec{t} := \begin{pmatrix} t \\ t \\ t \\ t \\ t \\ t \\ t \\ t \end{pmatrix}.$$

Candidate for odd p

$$V := \begin{array}{r} \alpha_3 \times \\ \alpha_2 \times \\ \alpha_2 \times \\ \alpha_2 \times \\ \alpha_1 \times \\ \alpha_1 \times \\ \alpha_1 \times \\ \alpha_0 \times \end{array} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \qquad \vec{t} := \begin{pmatrix} t \\ t \\ t \\ t \\ t \\ t \\ t \\ t \end{pmatrix}.$$

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- But we need a solution with all α's non-negative
- $M \in \mathbb{R}(t)^{k \times k}, \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$:

$$M \cdot \alpha = \begin{pmatrix} 1 + \varepsilon \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

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  - Let $\alpha' = M^{-1} \cdot e_1$
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  - $M \cdot \alpha = (1 + \varepsilon, 1, \cdots, 1)^T$

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Conclusions

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Thank you for your attention!