The Complexity of Simple and Optimal Deterministic Mechanisms for an Additive Buyer

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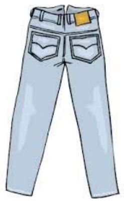
The Set-up

Seller has n items for sale

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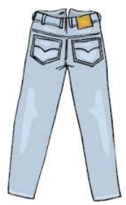    

Buyer has (private) value for each item

\$50 \$25 \$78 \$135 \$53

Probability distribution of value for each item, known to seller

F_1 F_2 F_3 F_4 F_5

Valuation of buyer drawn randomly from $F = F_1 \times F_2 \times \ldots \times F_n$

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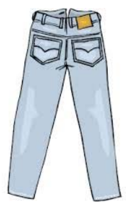




Buyer has (private) value for each item

$50 $25 $78 $135 $53

Additive buyer:
Value of a subset S of items = sum of values of items in S

The Set-up

- Seller can assign a price to each subset

 : $45  : $30 . . . .

  : $70 . . . .

. . . .

or offers a menu of only some subsets (bundles)

- Buyer's Utility for a subset S: u(S) = value(S) – price(S)
- Buyer buys subset S with maximum utility, if ≥ 0

(break ties say by highest value rule)

Optimal Pricing Problem

- Optimal Pricing (Revenue Maximization) Problem

  Find pricing that maximizes the expected revenue

$$\max E[\text{Revenue}] = \sum_{v \sim F} \Pr(v) \cdot \text{price}(S_v)$$

where S_v = bundle bought by buyer with valuation v

Single Item Pricing Scheme

- Set a price for each item

 : \$45  : \$30  : \$60  : \$150  : \$50

Price for each subset S : $\sum \{\text{price}(i) \mid i \in S\}$

- Optimal price for each item i :

value p_i^* that maximizes $p_i^* \cdot \Pr[value(i) \geq p_i^*]$

[Myerson ‘81]

Grand Bundle Pricing Scheme

- Can only buy the set of all items (the "grand bundle") for a given price, or nothing at all.

- There are examples where it gets more revenue than single item pricing:

 2 iid items with values {1, 2} with probability ½ each

 - Single item pricing: opt revenue 2 (eg. price 1 for each)

 - Grand bundle pricing: opt revenue 9/4

 price 3 for the grand bundle

Partition Pricing Scheme

- Partition the items into groups and assign price to each group in partition.


: $85


: $60


: $170

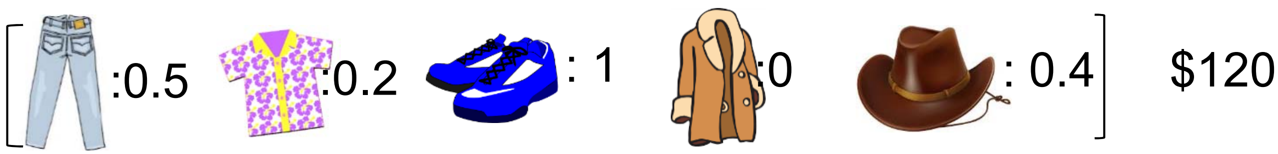
- Can buy any set of groups for sum of their prices

- Includes single item and grand bundle pricing as special cases
- Can get more revenue than both in some examples

Randomized Schemes (Lottery Pricing)

- Lottery = vector $(q_1,\ldots,q_n)$ of probabilities for the items
  If buyer buys the lottery then she gets each item i with probability q_i

- Lottery pricing: Menu = set of (lottery, price) pairs.



. . . . . .

- Buyer buys lottery with maximum expected utility

- There are examples where lottery pricing gives more revenue than the optimal deterministic pricing

Pricing schemes $\leftrightarrow$ Mechanism design

- Buyer submits a bid for each item
- Mechanism determines *allocation* the buyer receives and the *price* she pays
  Mechanism must be incentive compatible and individually rational
- Bundle pricings $\leftrightarrow$ deterministic mechanisms
- Lottery pricings $\leftrightarrow$ randomized mechanisms

Past Work

- Lots of work both in economic theory and in computer science
- 1 item: well-understood (also for many buyers) Myerson’81; randomization does not help
- 2 items: much more complicated; randomization can help

Work on

- Simple pricing schemes and their power/limitations
- Approximation of revenue
- Complexity

- Other models, e.g. unit-demand buyers, many buyers, correlated distributions

Past Work: Approximation

- Single item pricing: $\Theta(\log n)$ approximation to optimal revenue [Hart-Nisan'12, Li-Yao'13]
- Grand bundle: O(1) approximation for IID distributions [LY13]
- Better of single item/grand bundle: 6-approximation for any (independent) distributions [Babaioff et al'15]
- Approximation schemes for subclasses of distributions [Daskalakis et al '12, Cai-Huang'13]
- Reduction of many buyers to one, and O(1) approximation [Yao'15]

Past Work: Complexity

- Grand Bundle: Computing the best price for the grand bundle is #P-hard [Daskalakis et al ’12]
- Partition pricing: Computing the best partition and prices is NP-hard. But PTAS for best revenue achievable by any partition mechanism [Rubinstein ’16]
- Randomized mechanisms: #P-hard to compute the optimal solution/revenue [Daskalakis et al ’14]

Questions

- Is there an efficient algorithm that finds an optimal (deterministic) pricing?
- Is there such an algorithm when the instance has a “simple” optimal pricing?
- Is there a simple (i.e. easy to check) characterization of when single item pricing is optimal?
- For grand bundle pricing?

Results

- The optimal deterministic pricing problem is #P-hard, even if all distributions have support 2, and if the optimal is guaranteed to have a very simple form (we call it "discounted item pricing"): single item prices & price for grand bundle. Buyer can buy any subset for sum of its item prices or the grand bundle at its price
  - Also #P-hard to compute the optimal revenue.

- It is #P-hard to determine for a given instance
  - if single item pricing is optimal,
  - if grand bundle pricing is optimal

Results

- For IID distributions of support 2, the optimal revenue (even among randomized solutions) can be achieved by a discounted item pricing (i.e., single item prices & price for grand bundle), and it can be computed in polynomial time.

- For constant number of items (and any independent distributions), the problem can be also solved in polynomial time.

Integer Linear Program

- Let D_i= support of F_i and $D=D_1 \times \ldots \times D_n$ (exponential size)
- Variables: $x_{v,1},\ldots,x_{v,n} \in \{0,1\},\ \pi_v\ ,\ \forall v \in D$
- $(x_{v,1},\ldots,x_{v,n})$ = characteristic vector of bundle bought for valuation v, π_v its price

$$\max \sum_{v \in D} \pi_v \cdot \Pr[v]$$

Subject to

$$1.\ \forall v \in D:\ \ x_{v,i} \in \{0,1\}$$

$$2.\ \forall v \in D: \sum_{i \in [n]} v_i \cdot x_{v,i} - \pi_v \geq 0$$

$$3.\ \forall w,v \in D:\ \sum_{i \in [n]} w_i \cdot x_{w,i} - \pi_w \geq \sum_{i \in [n]} w_i \cdot x_{v,i} - \pi_v$$

(w does not envy the bundle of v)

- The LP ($x_{v,i} \in [0,1]$) models the optimal lottery problem

IID with support size 2

- Can assume wlog that support={1,b} with b>1

  (If support={0,b} then trivial: price all items at b. Otherwise rescale.)
- Let p = Pr(value=b), 1-p = Pr(value=1)
- Let $Q_i = \binom{n}{i} \cdot p^i (1-p)^{n-i}$ = Pr(i items at value b)
- Lemma: There exists an integer $k \in [0:n]$ such that

  $$(n-i)Q_i - (b-1)(Q_{i+1} + \ldots + Q_n)$$

  is < 0 for all $i : 0 \le i \le k$, and is $\ge$ 0 for all $i : k \le i \le n$.
- Optimal Pricing S*: Price every item at b, and offer the grand bundle at price $kb+n-k$

Proof Sketch

- Expected revenue of S* is $R^* = \sum_{1 \le i < k} bi \cdot Q_i + (kb + n - k) \sum_{k \le i \le n} Q_i$

- Since IID, the LP for the optimal lottery has a symmetric optimal solution [DW12], and the LP can be simplified to a more compact symmetric LP.

- Variables: x_i , $i=1,\ldots,n$: probability of getting a value b item when the valuation has i items at b

  y_i, $i=0,\ldots,n\text{-}1$: probability of getting a value 1 item when the valuation has i items at b

  π_i , $i=0,\ldots,n$: price of lottery for a valuation with i items at b

Proof Sketch ctd.

The symmetric LP maximizes $\sum_{i=0}^{n} \pi_i \cdot Q_i$

Relax the LP by keeping only some of the constraints

1. $0 \le x_i \le 1$ and $0 \le y_i \le 1$ for all i
2. $\pi_0 \le n y_0$ (the utility of the all-1 valuation is ≥ 0)
3. For each $i \in [n]$, the valuation w with $w_j = b$ for $j \le i$ and $w_j = 1$ for $j > i$ does not envy the lottery of the valuation v with $v_j = b$ for $j \le i$-1 and $w_j = 1$ for $j > i$-1

$$bix_i + (n-i)y_i - \pi_i \geq b(i-1)x_{i-1} + (n-i+b)y_{i-1} - \pi_{i-1}$$

- Combine the inequalities to upper bound every π_i in terms of the x, y variables

Proof Sketch ctd.

$\pi_0 \leq n y_0$

$\pi_i \leq b i x_i + (n-i) y_i - (b-1)(y_{i-1} + y_{i-2} + \ldots + y_1 + y_0)$

Replacing in the objective function every π_i by its upper bound

$\rightarrow$ linear form in x_i ,y_i that upper bounds optimal value

- Coefficient of x_i is biQ_i >0 $\Rightarrow$ expression maximized if x_i =1
- Coefficient of y_i is $(n\text{-}i)Q_i - (b\text{-}1)(Q_{i+1} + \ldots + Q_n)$, which is

  < 0 if $i < k$, and ≥ 0 if $i \geq k \Rightarrow$ expression maximized if

  we set y_i = 0 for all $i < k$ and y_i =1 for all $i \geq k$

- Substituting these values in the expression that upper bounds the objective function gives precisely R^*

#P-Hardness

- Reduction from the following problem, COMP.
  Input:
  1. Set B of integers $0<b_1 \leq b_2 \leq \ldots \leq b_n \leq 2^n$
  2. Subset $W \subset [n]$ of size $|W|=n/2$. Let $w=\Sigma_{i \in W} b_i$
  3. Integer t

Question: Is the number of subsets $S \subset [n]$ of size $|S|=n/2$ such that $\Sigma_{i \in S} b_i \geq w$ at least t ?

Construction

- n+1 items: n items $\leftrightarrow$ b_i's + special item
- First n items: almost iid with support {1, big}
  Item i: value 1 with probability $p=1/2(h+1)$, where $h=2^{2n}$
  value $h+1+b_i\delta$, where $\delta=1/2^{3n}$, with probability 1-p
- Item n+1: support $\{\sigma, \sigma+\alpha\}$, where $\sigma=1/p^n$, $\alpha=(n/2)h+w\delta$ ($<<\sigma$)
  value σ with probability $(\alpha/(\sigma+\alpha))+\varepsilon$ for some $\varepsilon=\varepsilon(t)=o(1/\sigma)$
  value $\sigma+\alpha$ with probability $(\sigma/(\sigma+\alpha))-\varepsilon$ (=almost 1)

Two Candidate Solutions

- Solution 1: Grand bundle at price $n+\sigma$ = sum of low values
  Equivalently, single item pricing with all prices= low values

- Solution 2: Discounted item pricing where all item prices=high values, and grand bundle price = $n + \sigma + \alpha$

Theorem: One of these two solutions is the unique optimal solution. #P-hard to tell which one of the two.

Solution 1 is optimal if the answer to the COMP question is No

$(\,|\, \{S \subset [n] \text{ of size } |S|=n/2 \text{ such that } \Sigma_{i \in S} b_i \geq w \}| < t)$

Solution 2 is optimal if the answer to the COMP question is Yes

$(\,|\, \{S \subset [n] \text{ of size } |S|=n/2 \text{ such that } \Sigma_{i \in S} b_i \geq w \}| \geq t)$

Two Candidate Solutions

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Corollaries:

1. #P-hard to tell if single item pricing is optimal
2. #P-hard to tell if grand bundle pricing is optimal

Proof Sketchy Outline

- Integer Linear Program, using the allocation variables $x_{v,i}$ and utility variables u_v instead of price variables π_v

$$(u_v = \sum_{i \in [n]} v_i \cdot x_{v,i} - \pi_v)$$

- Denote a valuation by (S,σ) (or (S,$\sigma+\alpha$)), for S$\subseteq$[n] if S=set of first n items that have high value and n+1th item has value σ (or $\sigma+\alpha$)

- In solution 1, all variables $x_{v,i}$ =1

$$\text{For } v = (S, \sigma),\ u_v = \sum_{i \in S} h_i$$

$$\text{For } v = (S, \sigma + \alpha),\ u_v = \alpha + \sum_{i \in S} h_i$$

Proof Sketchy Outline ctd.

- In Solution 2:

1. If $v = (S, \sigma + \alpha)$, all $x_{v,i} = 1$, $u_v = \sum_{i \in S} h_i$
2. If $v = (S, \sigma)$ and $\sum_{i \in S} h_i \geq \alpha$ then all $x_{v,i} = 1$, $u_v = \sum_{i \in S} h_i - \alpha$
3. If $v = (S, \sigma)$ and $\sum_{i \in S} h_i < \alpha$ then $x_{v,i} = 1$ for all $i \in S$,

   $x_{v,i} = 0$ for all $i \notin S$ and for $i = n+1$, and $u_v = 0$

- Every S with |S| > n/2 satisfies case 2,
- every S with |S| < n/2 satisfies case 3,
- a set S with |S| = n/2 satisfies case 2 if $\sum_{i \in S} b_i \geq w$

  and case 3 otherwise

Proof Sketchy Outline ctd.

- Relaxed ILP – keep only a subset of the envy constraints
  - $(S,\sigma+\alpha)$ does not envy $(\emptyset, \sigma+\alpha)$, for all $S \neq \emptyset$
  - $(\emptyset,\sigma+\alpha)$ does not envy (S,σ), and vice-versa, for all $S \subseteq [n]$,
  - for all $T \subset S \subset [n]$, (S,σ) does not envy (T,σ)

- Long sequence of lemmas shows that the optimal solution to the relaxed ILP must be either solution 1 or solution 2

- For $v=(\emptyset,\sigma+\alpha)$, if $x_{v,n+1} = 0$ then it must be Solution 1,

  if $x_{v,n+1} = 1$ then it must be Solution 2

Constant Number of Items

- #items =k =constant, support size m for each item
- V = set of m^k possible valuation vectors (polynomial)
- $d=2^k$ possible bundles (constant)

- Space R_+^d of possible price vectors p for the bundles partitioned by hyperplanes into cells such that $\forall$cell C $\forall$ valuation v buys the same bundle for all p$\in$C
- Hyperplanes:

$$\forall v \in V \ \forall j \in [d]: \sum_{l \in B_j} v_l - p_j = 0$$

$$\forall v \in V \ \forall j, j' \in [d]: \sum_{l \in B_j} v_l - p_j = \sum_{l \in B_{j'}} v_l - p_{j'}$$

$$\forall j, j' \in [d]: p_j = p_{j'}$$

Constant Number of Items

- The supremum revenue for price vectors in C is given by an LP, and is achieved at a vertex of C.

$\Rightarrow$ Optimum overall is achieved at a vertex of the subdivision

- Polynomial number of hyperplanes, constant dimension d
  $\Rightarrow$ polynomial number of vertices.
- Try them all and pick best.

Conclusions

- Showed that the optimal (deterministic) pricing problem is hard, and this holds even when the optimal solution is very simple : single item pricing + discount for grand bundle

- Can we find a polynomial time approximation scheme, or can we rule it out?

  When there is a simple optimal solution?

- IID case?

  Is there a PTAS?