Artificial Intelligence 1

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Problem Solving: Informed (Heuristic) Search

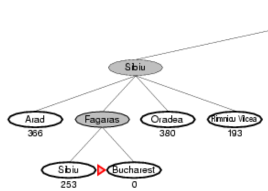
- **Uninformed (blind)** search algorithms can find an (optimal) solution to the problem, but they are usually not very efficient.
- **Informed (heuristic)** search algorithms can find solutions more efficiently thanks to exploiting problem-specific knowledge.
  - **How to use heuristics in search?**
    - BFS, A*, IDA*, RBFS, SMA*
  - **How to build heuristics?**
    - relaxation, pattern databases

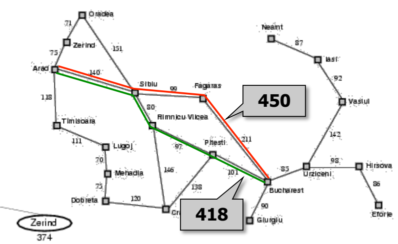
- Recall that we are looking for (the shortest) path from the initial state to some goal state.
- Which information can help the search algorithm?
  - For example, the length of path to some goal state.
  - However such information is usually not available (if it is available then we do not need to do search). Usually some **evaluation function f(n)** is used to evaluate „quality" of node n based on the length of path to the goal.
  - **best-first search**
    - The node with the smallest value of f(n) is used for expansion.
  - There are search algorithms with different views of f(n). Usually the part of f(n) is a **heuristic function h(n)** estimating the length of the shortest (cheapest) path to the goal state..
    - Heuristic functions are the most common form of additional information given to search algorithms
    - We will assume that **h(n) = 0 ⇔ n is goal**.

- Let us try to expand first the node that is closest to some goal state, i.e. f(n) = h(n).
  - **greedy best-first search algorithm**

Example (path Arad → Bucharest):
- We have a table of direct distances from any city to Bucharest.
- Note: this information was not part of the original problem formulation!

Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Dobreta	242	**Pitesti**	100
Eforie	161	**Rimmicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

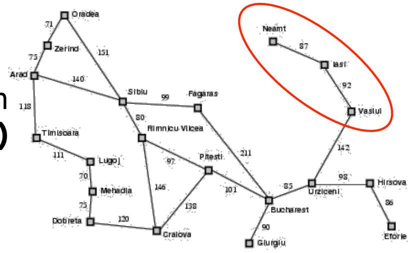





Nejkratší cesta?

- We already know that the greedy algorithm **may not find the the optimal path**.
- **Can we at least always find some path?**
  - If we expand first the node with the smallest cost then the algorithm **may not find any solution**.

    Example: path Iasi → Fagaras
    - Go to Neamt, then back to Iasi, Neamt, …
    - We need to detect repeated visits in cities!



- **Time complexity O(b^m^),** where m is the maximal depth
- **Memory complexity O(b^m^)**
- A good heuristic function can significantly decrease the practical complexity.

- Let us now try to use **f(n) = g(n) + h(n)**
  - Recall that g(n) is he cost of path from root to n
  - probably the most popular heuristic search algorithm
  - f(n) represents the cost of path through n
  - the algorithm does not extend already long paths

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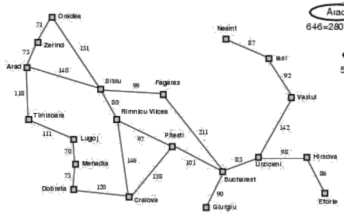
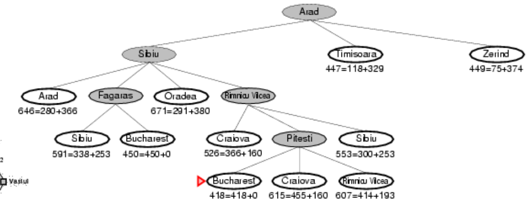
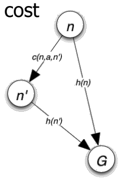




What about completeness and optimality of A*?

First a few definitions:

- **admissible heuristic h(n)**
  - **h(n) ≤ „the cost of the cheapest path from n to goal "**
  - an optimistic view (the algorithm assumes a better cost then the real one)
  - function f(n) in A* is a lower estimate of the cost of path through n
- **monotonous (consistent) heuristic h(n)**
  - let n' be a successor of n via action a and c(n,a,n') be the transition cost
  - **h(n) ≤ c(n,a,n') + h(n')**
  - this is a form of triangle inequality



Monotonous heuristic is admissible.

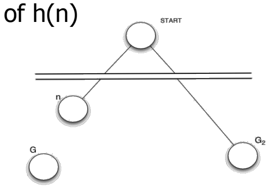
let $n_1, n_2, \ldots, n_k$ be the optimal path from n_1 to goal n_k, then
$h(n_i) - h(n_{i+1}) \leq c(n_i,a_i,n_{i+1})$, via monotony
$h(n_1) \leq \Sigma_{i=1,..,k-1}\, c(n_i,a_i,n_{i+1})$, after „sum"

For a monotonous heuristic the values of f(n) are not decreasing over any path.

Let n' be a successor of n, i.e. g(n') = g(n) + c(n,a,n'), then
f(n') = g(n') + h(n') = g(n) + c(n,a,n') + h(n') ≥ g(n) + h(n) = f(n)

- **If h(n) is an admissible heuristic then the algorithm A* in TREE-SEARCH is optimal.**
  - in other words – the first expanded goal is optimal
  - Let G_2 be sub-optimal goal from the fringe and C* be the optimal cost
    - $f(G_2) = g(G_2) + h(G_2) = g(G_2) > C^*$, because $h(G_2) = 0$
  - Let n be a node from the fringe and being on the optimal path
    - $f(n) = g(n) + h(n) \leq C^*$, via admissibility of h(n)
  - together
    - $f(n) \leq C^* < f(G_2)$,

    i.e., the algorithm must expand n before G_2 and this way it finds the optimal path.

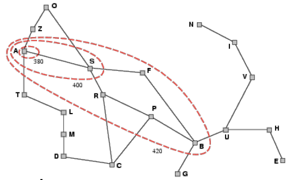


Algorithm A*: optimality

- **If h(n) is a monotonous heuristic then the algorithm A* in GRAPH-SEARCH is optimal.**
  - Possible problem: reaching the same state for the second time using a better path – classical GRAPH-SEARCH ignores this second path!
  - A possible solution: selection of better from both paths leading to a close node (extra bookkeeping) or using monotonous heuristic.
    - for monotonous heuristics, the values of f(n) are not decreasing over any path
    - A* selects for expansion the node with the smallest value of f(n), i.e., the values f(m) of other open nodes m are not smaller, i.e., among all "open" paths to n there cannot be a shorter path than the path just selected (no path can shorten)
    - hence, the first closed goal node is optimal

Algorithm A*: properties

- For non-decreasing function f(n) we can draw **contours** in the state graph (the nodes inside a given contour have f-costs less than or equal to the contour value.
  - for h(n) = 0 we obtain circles around the start
  - for more accurate h(n) we use, the bands will stretch toward the goal state and become more narrowly focused around the optimal path.



  - A* expands all nodes such that f(n) < C* on the contour
  - A* can expand some nodes such that f(n) = C*
  - the nodes n such that f(n) > C* are never expanded
  - the algorithm A* is **optimality efficient** for any given consistent heuristic

Time complexity:
A* can expand an exponential number of nodes
- this can be avoided if |h(n)-h*(n)| ≤ O(log h*(n)), where h*(n) is the cost of optimal path from n to goal

Space complexity:
A* keeps in memory all expanded nodes

A* usually runs out of space long before it runs out out of time

Iterative-deepening A*

- A simple way to decrease memory consumption is iterative deepening.
- **Algorithm IDA***

```
function IDA*(problem) returns a solution sequence
   inputs: problem, a problem
   static: f-limit, the current f- COST limit
           root, a node

   root ← MAKE-NODE(INITIAL-STATE[problem])
   f-limit ← f- COST(root)
   loop do
      solution, f-limit ← DFS-CONTOUR(root, f-limit)
      if solution is non-null then return solution
      if f-limit = ∞ then return failure; end

function DFS-CONTOUR(node, f-limit) returns a solution sequence and a new f- COST limit
   inputs: node, a node
           f-limit, the current f- COST limit
   static: next-f, the f- COST limit for the next contour, initially ∞

   if f- COST[node] > f-limit then return null, f- COST[node]
   if GOAL-TEST[problem](STATE[node]) then return node, f-limit
   for each node s in SUCCESSORS(node) do
      solution, new-f ← DFS-CONTOUR(s, f-limit)
      if solution is non-null then return solution, f-limit
      next-f ← MIN(next-f, new-f); end
   return null, next-f
```

- the search limit is defined using the cost f(n) instead of depth
- for the next iteration we use the smallest value f(n) of node n that exceeded the limit in the last iteration
- frequently used algorithm

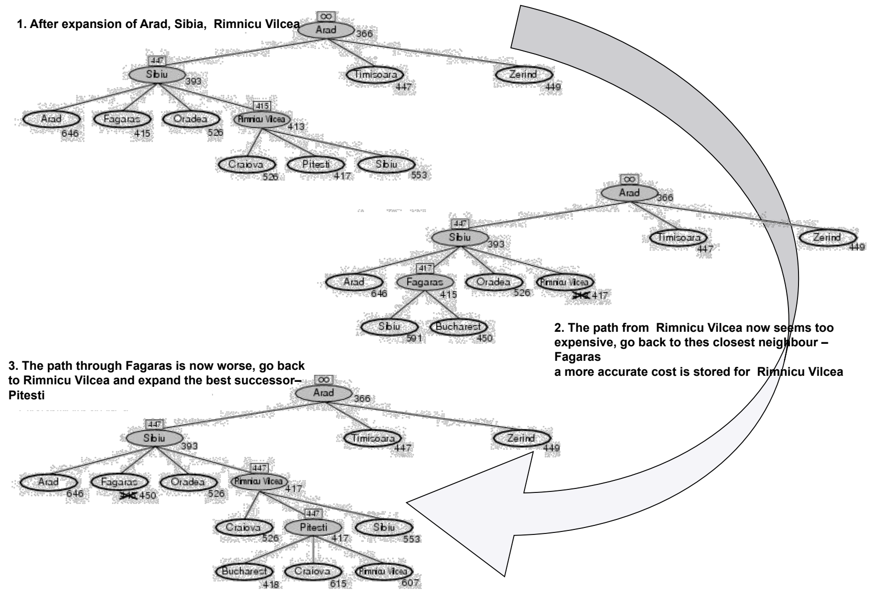
Recursive best-first search

- Let us try to mimic standard best-first search, but using only linear space
  - the algorithm stops exploration if there is an alternative path with better cost f(n)
  - when the algorithm goes back to node n, it replaces the value f(n) using the cost of successors (remembers the best leaf in the forgotten subtree)
- **If h(n) is an admissible heuristic then the algorithm is optimal.**
- **Space complexity O(bd)**
- **Time complexity is still exponential** (suffers from excessive node re-generation)

```
function RECURSIVE-BEST-FIRST-SEARCH(problem) returns a solution, or failure
   RBFS(problem, MAKE-NODE(INITIAL-STATE[problem]), ∞)

function RBFS(problem, node, f_limit) returns a solution, or failure and a new f-cost limit
   if GOAL-TEST[problem](STATE[node]) then return node
   successors ← EXPAND(node, problem)
   if successors is empty then return failure, ∞
   for each s in successors do
      f[s] ← max(g(s) + h(s), f[node])
   repeat
      best ← the lowest f-value node in successors
      if f[best] > f_limit then return failure, f[best]
      alternative ← the second-lowest f-value among successors
      result, f[best] ← RBFS(problem, best, min(f_limit, alternative))
      if result ≠ failure then return result
```

Recursive BFS - example



Simplified memory-bounded A*

- IDA* and RBFS do not exploit availably memory!
- This is a pity as the already expanded nodes are re-expanded again (waste of time)
- Let us try to modify classical A*

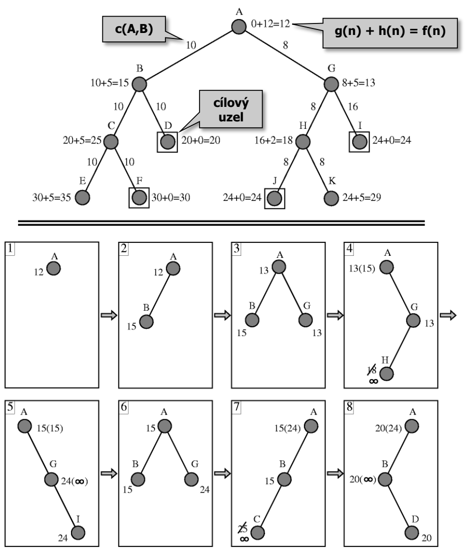
```
function SMA*(problem) returns a solution sequence
  inputs: problem, a problem
  static: Queue, a queue of nodes ordered by f-cost

  Queue ← MAKE-QUEUE({MAKE-NODE(INITIAL-STATE[problem])})
  loop do
     if Queue is empty then return failure
     n ← deepest least-f-cost node in Queue
     if GOAL-TEST(n) then return success
     s ← NEXT-SUCCESSOR(n)
     if s is not a goal and is at maximum depth then
        f(s) ← ∞
     else
        f(s) ← MAX(f(n), g(s)+h(s))
     if all of n's successors have been generated then
        update n's f-cost and those of its ancestors if necessary
     if SUCCESSORS(n) all in memory then remove n from Queue
     if memory is full then
        delete shallowest, highest-f-cost node in Queue
        remove it from its parent's successor list
        insert its parent on Queue if necessary
     insert s on Queue
  end
```

Path from root to this non-goal node can be stored in memory, hence no optimal path through this node can be found.

  - when memory is full, drop the worst leaf node – the node with the highest f-value (if there are such nodes then drop the shallowest node)
  - similarly to RBFS back up the value of the forgotten node to its parent

Simplified memory-bounded A* - example

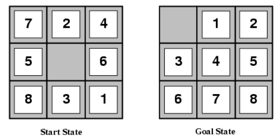


- Assume memory for **three nodes** only.
- If there is enough memory to store an optimal path then SMA* find optimal solution.
- Otherwise it finds the best path with available memory.
  - If the cost of J would be 19, then this is optimal goal, but the path to it can be stored in memory!

Looking for heuristics

How to find admissible heuristics?

Example: 8-puzzle



- 22 steps to goal in average
- branching factor around 3
- (complete) search tree: $3^{22} \approx 3{,}1 \times 10^{10}$ nodes
- the number of reachable states is only 9!/2 = 181 440
- for 15-puzzle there are 10^{13} states
- We need some heuristic, preferable admissible
  - h_1 = „the number of misplaced tiles“
    = 8
  - h_2 = „the sum of the distances of the tiles from the goal positions“
    = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18
    a so called Manhattan heuristic
  - the optimal solution needs 26 steps

How to characterize the quality of a heuristic?

Effective branching factor b*

- Let the algorithm needs N nodes to find a solution in depth d
- b* is a branching factor of a uniform tree of depth d containing N+1 nodes

  $N+1 = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d$

Example:

- 15-puzzle
- the average over 100 instances for each of various solution lengths

	Search Cost			Effective Branching Factor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	3644035	227	73	2.78	1.42	1.24
14	–	539	113	–	1.44	1.23
16	–	1301	211	–	1.45	1.25
18	–	3056	363	–	1.46	1.26
20	–	7276	676	–	1.47	1.27
22	–	18094	1219	–	1.48	1.28
24	–	39135	1641	–	1.48	1.26

- **Is h_2** (from 8-puzzle) **always better than h_1 and how to recognize it?**
  - notice that $\forall n\ h_2(n) \geq h_1(n)$
  - We say that **h_2 dominates h_1**
  - A* with h_2 never expands more nodes than A* with h_1
    - A* expands all nodes such that f(n) < C*, tj. h(n) < C* - g(n)
    - In particular if it expands a node using h_2, then the same node must be expanded using h_1
- **It is always better to use a heuristic function giving higher values provided that**
  - **the limit C* - g(n) is not exceeded** (then the heuristic would not be admissible)
  - **the computation time is no too long**

Can an agent construct admissible heuristics for any problem?

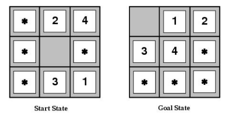
Yes via **problem relaxation**!

- relaxation is a simplification of the problem such that the solution of the original problem is also a solution of the relaxed problem (even if not necessarily optimal)
- we need to be able to solve the relaxed problem fast
- the cost of optimal solution to a relaxed problem is a lower bound for the solution to the original problem and hence it is an admissible (and monotonous) heuristic for the original problem

- **Example (8-puzzle)**
  - A tile can move from square A to square B if:
    - A is horizontally or vertically adjacent to B
    - B is blank
  - possible relaxations (omitting some constraints to move a tile):
    - a tile can move from square A to square B if A is adjacent to B (Manhattan distance)
    - a tile can move from square A to square B if B is blank
    - a tile can move from square A to square B (heuristic h_1)

Another approach to admissible heuristics is using a **pattern database**

- based on solution of specific sub-problems (patterns)
- by searching back from the goal and recording the cost of each new **pattern** encountered



- heuristic is defined by taking the worst cost of a pattern that matches the current state
- Beware! The "sum" of costs of matching patterns need not be a admissible (the steps for solving one pattern may be used when solving another pattern).

If there are **more heuristics**, we can always use the **maximum** value from them (such a heuristic dominates each of used heuristics).