

Sets

- A **Set** is a collection of objects.
- An object 'a' that belongs to a set S is a **member** or **element** of S and we write $a \in S$.
- If 'a' is not a member of set S, we write $a \notin S$.

Special Sets:

- \mathbb{N} - the set of **Natural Numbers**:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

- \mathbb{P} - the set of **Positive Integers**:

$$\mathbb{P} = \{1, 2, 3, 4, \dots\}$$

- \mathbb{Z} - the set of **all Integers**:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

- \mathbb{Q} - the set of **Rational Numbers**:

$$\mathbb{Q} = \{\text{numbers of the form } m/n \text{ where } m \in \mathbb{Z}, n \in \mathbb{Z}, \text{ and } n \neq 0\}$$

- \mathbb{R} - the set of **Real Numbers**:

$$\mathbb{R} = \{\text{numbers - rational or not}\}$$

- Two sets are **equal** if they contain the same elements:

$$\{2,4,6,8,10\} = \{10,8,6,4,2\} = \{2,8,2,6,2,10,4,2\}$$

Note: The order of the elements is not important.

- Sets are often described by properties of their elements:

$$\{n : n \in \mathbb{N} \text{ and } n \text{ is even}\} = \{n \in \mathbb{N} : n \text{ is even}\}$$

$$\{x : x \in \mathbb{R} \text{ and } 1 \leq x < 3\} = \{x \in \mathbb{R} : 1 \leq x < 3\}$$

$$\{n^2 : n \in \mathbb{N}\} = \{0,1,4,9,16,25,36,\dots\}$$

$$\{(-1)^n : n \in \mathbb{N}\} = \{-1,1\}$$

- S is a **subset** of T provided that every element of S belongs to T and we write $S \subseteq T$.

$$\mathbb{P} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

$$S \subseteq S$$

- Two sets S and T are **equal** iff $S \subseteq T$ and $T \subseteq S$.
- T is a **proper subset** of S, written as $T \subset S$, if $T \subseteq S$ and $T \neq S$.

- **Intervals:**

Brackets signify **closed** intervals.

Parentheses signify **open** intervals.

Examples: with $a, b \in \mathbb{R}$ and $a < b$:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$

$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

- The set that contains zero elements is called the **Empty Set** or **Null Set** and is written as either $\{ \}$ or \emptyset :

$$\{n \in \mathbb{N} : 2 < n < 3\}$$

$$\{x \in \mathbb{R} : x^2 < 0\}$$

- Sets can be members of other sets:

$\{\{1, 2\}, \{1, 3\}, \{2\}, \{3\}\}$ - set of 4 members

$\{\emptyset\}$ - set of 1 member

$\{ \} = \emptyset$ - set of 0 members

Note: $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$, but $\emptyset \notin \emptyset$.

● $\mathcal{P}(S)$ - The **Power Set of S**:

$$\mathcal{P}(S) = \{\text{all subsets of } S\}$$

- with $S = \emptyset$

$$\mathcal{P}(S) = \{\emptyset\}$$

- with $S = \{a\}$

$$\mathcal{P}(S) = \{\emptyset, \{a\}\}$$

- with $S = \{a,b\}$

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

- with $S = \{a,b,c\}$

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

- Σ - an **alphabet**:

Σ is a finite, nonempty set whose members are symbols called **letters**.

- **Word**:

any finite **string of letters** from Σ

- λ - the **Null Word** or **Empty Word**:

λ = the string with no letters.

- Σ^* - a **language** over Σ :

$\Sigma^* = \{ \text{all words using letters from } \Sigma \}$

- with $\Sigma = \{a\}$

$\Sigma^* = \{ \lambda, a, aa, aaa, aaaa, aaaaa, \dots \}$

- with $\Sigma = \{a, b\}$

$\Sigma^* = \{ \lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, \dots \}$

- with $\Sigma = \{0, 1, 2\}$

$\Sigma^* = \{ \lambda, 0, 1, 2, 00, 01, 02, 10, 11, 12, 20, 21, 22, 000, \dots \}$

- with $\Sigma = \{a, b\}$ and $A = \{w \in \Sigma^* : \text{length}(w) = 2\}$

$A = \{aa, ab, ba, bb\}$

- with $\Sigma = \{a, b\}$ and $B = \{w \in \Sigma^* : \text{length}(w) \text{ is even}\}$

$B = \{ \lambda, aa, ab, ba, bb, aaaa, aaab, aaba, aabb, \dots \}$

Set Operations

- **Union:**

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}$$

- **Intersection:**

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Note: A and B are **disjoint** if $A \cap B = \emptyset$.

- **Relative Complement:**

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

- **Symmetric Difference or Exclusive Or:**

$$A \oplus B = \{x : x \in A \text{ or } x \in B \text{ but not both}\}$$

$$\text{Note: } A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

- For $A = \{n \in \mathbb{N} : n \leq 11\}$ and $B = \{n \in \mathbb{N} : n \text{ is even and } n \leq 18\}$,

$$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 18\}$$

$$A \cap B = \{0, 2, 4, 6, 8, 10\}$$

$$A \setminus B = \{1, 3, 5, 7, 9, 11\}$$

$$B \setminus A = \{12, 14, 16, 18\}$$

$$A \oplus B = \{1, 3, 5, 7, 9, 11, 12, 14, 16, 18\}$$

- **U - Universal Set or Universe:**

U = A fixed set such that we consider only its elements and subsets.

- **A^c - the complement or absolute complement of A:**

For $A \subseteq U$, $A^c = U \setminus A$

- For $U = \mathbb{N}$, $A = \{n \in \mathbb{N} : n \leq 11\}$, and $E = \{n \in \mathbb{N} : n \text{ is even}\}$,

$$A^c = \{n \in \mathbb{N} : n \geq 12\}$$

$$E^c = \{n \in \mathbb{N} : n \text{ is odd}\}$$

Laws of Algebra of Sets

- **Commutative Laws:**

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- **Associative Laws:**

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

- **Distributive Laws:**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- **Idempotent Laws:**

$$A \cup A = A$$

$$A \cap A = A$$

- **Identity Laws:**

$$A \cup \emptyset = A$$

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

$$A \cap U = A$$

● **Double Complementation:**

$$(A^c)^c = A$$

● **Complementation Laws:**

$$A \cup A^c = U$$

$$A \cap A^c = \emptyset$$

$$U^c = \emptyset$$

$$\emptyset^c = U$$

● **DeMorgan's Laws:**

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Example: Prove $(A \cup B) \cap A^c \subseteq B$

$(A \cup B) \cap A^c$	$= A^c \cap (A \cup B)$	commutative law
	$= (A^c \cap A) \cup (A^c \cap B)$	distributive law
	$= (A \cap A^c) \cup (A^c \cap B)$	commutative law
	$= \emptyset \cup (A^c \cap B)$	complementation law
	$= (A^c \cap B) \cup \emptyset$	commutative law
	$= (A^c \cap B)$	identity law

Therefore, $(A \cup B) \cap A^c \subseteq B$ since if $x \in (A^c \cap B)$ then $x \in B$.

Ordered Pairs

Consider sets S and T with $s \in S$ and $t \in T$:

- (s,t) - **an ordered pair**:

(s,t) is an ordered pair where s is the first element and t is the second element and **the order is important**.

Note: $(s_1,t_1) = (s_2,t_2)$ iff $s_1 = s_2$ and $t_1 = t_2$.

- $S \times T$ - the **product** of S and T :

$$S \times T = \{(s,t) : s \in S \text{ and } t \in T\}$$

Note: $S \times S = S^2$

- For $S = \{1,2,3,4\}$ and $T = \{a,b,c\}$,

$$S \times T = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c), \\ (4,a),(4,b),(4,c)\}$$

- $|S|$ = the number of elements in the set S

$$|\emptyset| = 0$$

$$|\{1,2,3,\dots,n\}| = n \text{ for } n \in \mathbb{P}$$

$$|S \times T| = |S| \cdot |T|$$

$$|\mathcal{P}(S)| = 2^{|S|}$$

Functions

- A **function** f **defined on** S with **values in** T assigns to each element $x \in S$ a unique element $f(x) \in T$.
- S is called the **domain** of f - $\text{Dom}(f)$.
- $f(x)$ is called the **image** of x under f .
- $\text{Im}(f)$ - **image of f** :

$$\text{Im}(f) = \{f(x) : x \in \text{Dom}(f)\}$$

- a **codomain** of f is any set containing $\text{Im}(f)$.
- $f: S \rightarrow T$

this denotes a function f with domain S and codomain T .

- Example:

$$S = \{-2, -1, 1, 2, 3\}; T = \{0, 1, 4, 9, 16, 25\}$$

$$f: S \rightarrow T \text{ such that } f(x) = x^2$$

$$f(-2) = 4; f(-1) = 1; f(1) = 1; f(2) = 4; f(3) = 9$$

$$\text{Im}(f) = \{1, 4, 9\}$$

Graph of a Function

- Consider a function $f: S \rightarrow T$

$$\text{Graph}(f) = \{(x,y) \in S \times T : y = f(x)\}$$

- Example:

Using the above sets S and T and function f with $f(x) = x^2$

$$\text{Graph}(f) = \{(-2,4), (-1,1), (1,1), (2,4), (3,9)\}$$

- More precise definition of a function:

A **function** with domain S and codomain T is a subset G of $S \times T$ satisfying:

for each $x \in S$ there is exactly one $y \in T$ such that $(x,y) \in G$

- **One-to-One Function:**

A function $f: S \rightarrow T$ is called one-to-one in case distinct elements in S have distinct images in T under f :

if $x_1, x_2 \in S$ and $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

or

if $x_1, x_2 \in S$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$

Given $f: S \rightarrow T$, we say f is **one-to-one** iff for each $y \in T$ there is **at most one** $x \in S$ such that $(x, y) \in G$.

- Example:

$$S = \{-2, -1, 3, 5\}$$

$$T = \{0, 1, 4, 9, 16, 25\}$$

$$f: S \rightarrow T \text{ such that } f(x) = x^2$$

- **Onto Function:**

Given $f: S \rightarrow T$, we say f maps **onto** T iff for each $y \in T$ there is **at least** one $x \in S$ such that $(x,y) \in G$.

- Example:

$$S = \{-2, -1, 1, 2, 3\}$$

$$T = \{1, 4, 9\}$$

$$f: S \rightarrow T \text{ such that } f(x) = x^2$$

- **One-to-One Correspondence:**

A function that is both one-to-one and maps onto.

Given $f: S \rightarrow T$, we say f is a **one-to-one correspondence** between S and T iff for each $y \in T$ there is **exactly** one $x \in S$ such that $(x,y) \in G$.

- Example:

$$S = \{-4, -2, -1, 0, 3, 5\}$$

$$T = \{0, 1, 4, 9, 16, 25\}$$

$$f: S \rightarrow T \text{ such that } f(x) = x^2$$

Examples:

- Define $f: \mathbb{N} \rightarrow \mathbb{N}$ by the rule $f(n) = 2n$

- f is one-to-one since

$$f(n_1) = f(n_2) \text{ implies } 2n_1 = 2n_2 \text{ implies } n_1 = n_2$$

- f does not map onto \mathbb{N}

since $\text{Im}(f)$ consists only of the even natural numbers.

- Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by the rule $f(x) = 3x - 5$

- f is one-to-one since

$$f(x_1) = f(x_2) \text{ implies } 3x_1 - 5 = 3x_2 - 5 \text{ implies } x_1 = x_2$$

- f maps onto \mathbb{R}

for each $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$; namely, $x = (y+5)/3$.

- f is a one-to-one correspondence between \mathbb{R} and \mathbb{R}

because it is both one-to-one and maps onto \mathbb{R}

Special Functions

- 1_S - **Identity Function on S:**

the function that maps each element of S onto itself

$$1_S(x) = x \text{ for all } x \in S$$

1_S is a one-to-one correspondence of S onto S

- **Constant Function:**

With $y_0 \in T$, $f(x) = y_0$ for all $x \in S$

- χ_A - **Characteristic Function of A (where $A \subseteq S$):**

$$\chi_A = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \in S \setminus A \end{cases}$$

- **Composition:**

Consider $f: S \rightarrow T$ and $g: T \rightarrow U$, the **composition** $g \circ f: S \rightarrow U$ is given by the rule

$$g \circ f(x) = g(f(x)) \quad \text{for all } x \in S$$

- If $f(x) = x^3 + 2x$ and $g(x) = x^7$ for $x \in \mathbb{R}$,

then $h(x) = g(f(x)) = g(x^3 + 2x) = (x^3 + 2x)^7$ for $x \in \mathbb{R}$.

- **Associativity of Composition:**

If $f: S \rightarrow T$, $g: T \rightarrow U$, and $h: U \rightarrow V$, then

$$h \circ (g \circ f) = (h \circ g) \circ f = h \circ g \circ f$$

- **Example:**

If $f(x) = x^4$, $g(y) = (y^2 + 1)^{1/2}$, and $h(z) = z^2 + 72$, then

$$\begin{aligned} h \circ (g \circ f)(x) &= h(g \circ f(x)) \\ &= h(g(f(x))) \\ &= h(g(x^4)) \\ &= h((x^8 + 1)^{1/2}) \\ &= ((x^8 + 1)^{1/2})^2 + 72 \\ &= x^8 + 73 \end{aligned}$$

$$\begin{aligned} (h \circ g) \circ f(x) &= (h \circ g)(f(x)) \\ &= h(g(f(x))) \\ &= x^8 + 73 \end{aligned}$$

- **Example:**

If $f(x) = x^4$ for $x \in [0, \infty)$, $g(x) = (x + 2)^{1/2}$ for $x \in [0, \infty)$, and $h(x) = x^2 + 1$ for $x \in \mathbb{R}$, then

$$\begin{aligned} h \circ g \circ f(x) &= h(g(f(x))) = h((x^4 + 2)^{1/2}) = (x^4 + 2) + 1 \\ &= x^4 + 3 \text{ for } x \in [0, \infty) \end{aligned}$$

$$\begin{aligned} f \circ g \circ h(x) &= f(g(h(x))) = f((x^2 + 1 + 2)^{1/2}) \\ &= (x^2 + 3)^2 \text{ for } x \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} f \circ h \circ g(x) &= f(h((x + 2)^{1/2})) = f(x + 2 + 1) \\ &= (x + 3)^4 \text{ for } x \in [0, \infty) \end{aligned}$$

Inverses of Functions

- An **inverse** of a function $f: S \rightarrow T$ is a function $f^{-1}: T \rightarrow S$ such that $f^{-1} \circ f = 1_S$ and $f \circ f^{-1} = 1_T$; that is,

$$f^{-1}(f(x)) = x \quad \text{for all } x \in S$$

and

$$f(f^{-1}(y)) = y \quad \text{for all } y \in T$$

- Functions that have inverses are called **invertible**.

- Theorem:

The function $f: S \rightarrow T$ is invertible iff f is one-to-one and maps S onto T (f is a one-to-one correspondence).

- $f^{-1}(y) =$ that unique $x \in S$ such that $f(x) = y$

- Example:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^3 + 1$

- f is one-to-one since

$$f(x_1) = f(x_2) \quad \text{implies } (x_1)^3 + 1 = (x_2)^3 + 1$$

$$\text{implies } (x_1)^3 = (x_2)^3$$

implies $x_1 = x_2$ (since cube root is unique)

- f maps \mathbb{R} onto \mathbb{R} since

for each $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$; namely, $x = (y - 1)^{1/3}$.

Note: $f^{-1}(y) = (y - 1)^{1/3}$

Consider $f: S \rightarrow T$, $A \subseteq S$, and $B \subseteq T$

- $f(A)$ - the **image of the set A under f**:

$$f(A) = \{f(x) : x \in A\}$$

- $f^{-1}(y)$ - the **pre-image of the element y under f**

$$f^{-1}(y) = \{x \in S : f(x) = y\} \quad \text{for } y \in T$$

- $f^{-1}(y)$ is the **solution set** for the equation $f(x) = y$.

- $f^{-1}(B)$ - the **pre-image of a set B under f**

$$f^{-1}(B) = \{x \in S : f(x) \in B\}$$

- if f is invertible:

$$f^{-1}(B) = \{f^{-1}(y) : y \in B\} = f^{-1}(B)$$

- Example:

Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$

$$- f^{-1}(4) = \{x \in \mathbb{R} : x^2 = 4\} = \{-2, 2\}$$

$$- f^{-1}([1, 9]) = \{x \in \mathbb{R} : x^2 \in [1, 9]\} = \{x \in \mathbb{R} : 1 \leq x^2 \leq 9\} \\ = [-3, -1] \cup [1, 3]$$

$$- f^{-1}([-1, 0]) = \{x \in \mathbb{R} : x^2 \in [-1, 0]\} = \{x \in \mathbb{R} : -1 \leq x^2 \leq 0\} \\ = \{0\}$$

● Example:

Consider the function $g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $g(m,n) = m^2 + n^2$

- $g^{-1}(0) = \{(0,0)\}$
- $g^{-1}(1) = \{(0,1), (1,0)\}$
- $g^{-1}(2) = \{(1,1)\}$
- $g^{-1}(3) = \emptyset$
- $g^{-1}(4) = \{(0,2), (2,0)\}$
- $g^{-1}(25) = \{(0,5), (3,4), (4,3), (5,0)\}$

● Example:

Consider $h: \mathbb{Z} \rightarrow \{-1, 1\}$ where $h(n) = (-1)^n$

- $h^{-1}(1) = \{n \in \mathbb{Z} : n \text{ is even}\}$
- $h^{-1}(-1) = \{n \in \mathbb{Z} : n \text{ is odd}\}$

Note: $h^{-1}(1) \cup h^{-1}(-1) = \mathbb{Z}$

Sequences

● Summation:

$$\sum_{k=1}^{10} k^2 = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385$$

$$\sum_{k=1}^n k^2 = 1 + 4 + 9 + 16 + \dots + n^2$$

● Products:

$$\prod_{k=1}^4 k^2 = 1 \cdot 4 \cdot 9 \cdot 16 = 576$$

$$\prod_{k=1}^n k^2 = 1 \cdot 4 \cdot 9 \cdot \dots \cdot n^2$$

● n! - n factorial:

For $n \in \mathbb{P}$ the product of the first n integers is called $n!$

$$n! = \prod_{k=1}^n k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$$

$1! = 1$; $2! = 2$; $3! = 6$; $4! = 24$; $5! = 120$; $0! = 1$ by definition.

- **Sequences**

A sequence on \mathbb{N} is a list $s_0, s_1, s_2, \dots, s_n, \dots$ that has a specified value s_n for each integer $n \in \mathbb{N}$.

s_n (or $s(n)$) is called the n th **term** of the sequence.

The sequence can be denoted by (s_n) or $(s_n)_{n \in \mathbb{N}}$ or (s_0, s_1, s_2, \dots) .

A sequence (s_n) is a function whose domain is either \mathbb{N} or $\{m, m+1, m+2, \dots\}$ for some integer m .

- Example:

Let $s_n = n!$

$$(s_n) = (1, 1, 2, 6, 24, \dots)$$

The *set of values* is $\{1, 2, 6, 24, \dots\}$

- Example:

Let $a_n = (-1)^n$ for $n \in \mathbb{N}$

$$(a_n) = (1, -1, 1, -1, 1, -1, 1, \dots)$$

The *set of values* is $\{-1, 1\}$

- Example:

Let $b_n = 1/n^2$

The domain must clearly avoid the case $n = 0$

$$(b_n) = (1, 1/4, 1/9, 1/16, \dots)$$

- Example:

Let Σ be an alphabet.

For each $k \in \mathbb{N}$, $\Sigma^k = \{w \in \Sigma^* : \text{length}(w) = k\}$

$(\Sigma^k)_{k \in \mathbb{N}}$ is a sequence of subsets of Σ^*

$$\Sigma^0 = \{\lambda\}$$

$$\Sigma^1 = \Sigma$$

$$\bigcup_{k \in \mathbb{N}} \Sigma^k = \Sigma^*$$

- Notation:

$$\bigcup_{k \in \mathbb{N}} A_k = \{x : x \in A_k \text{ for at least one } k \in \mathbb{N}\}$$

$$\bigcap_{k \in \mathbb{N}} A_k = \{x : x \in A_k \text{ for all } k \in \mathbb{N}\}$$

- **Finite Sequences:**

A **finite sequence** is a string of objects that are listed using subscripts from a finite subset of \mathbb{Z} of the form $\{m, m+1, \dots, n\}$.