Elementary Logic

- **Propositions:**

  A **proposition** is any statement that is either **true** or **false.**

- Examples - The following are propositions:

  - Julius Caesar was president of the United States.
  - $2 + 2 = 4$.
  - $2 + 3 = 7$.
  - If a set has n elements, then it has 2^n subsets.
  - $x + y = y + x$ for all $x, y \in \mathbb{R}$.
  - $2^n = n^2$ for some $n \in \mathbb{N}$.
  - If the world is flat, then $2 + 2 = 4$.

- Examples - The following are not propositions:

  - Why is induction important?
  - Go directly to jail.
  - Either that dog goes or I do.
    (truth value not generally known)
  - $x - y = y - x$.
    (domain of the symbols is not specified)

- Examples - Ambiguous propositions:

  - Teachers are overpaid.
  - CIS 11 is fun.

Compound Propositions

- Simple propositions can be combined, using standard connective symbols, to obtain compound propositions.

- **Standard Connective Symbols**

  $\neg$ 'not' or negation
  $\wedge$ 'and'
  $\vee$ 'or' (inclusive)
  $\rightarrow$ 'implies' or the conditional implication
  $\leftrightarrow$ 'if and only if' or the biconditional
  $\oplus$ 'xor' or exclusive or

- Examples:
  - The number 4 is positive and the number 3 is negative.
    Let p = the number 4 is positive.
    Let q = the number 3 is negative.
    The compound proposition can be written as $p \wedge q$.

  - If the world is flat, then 2 + 2 = 4.
    Let r = the world is flat.
    Let s = 2+ 2 = 4.
    The compound proposition can be written as $r \rightarrow s$.

  - It is not true that 3 is an even number or 7 is a prime.
    Let p = 3 is an even number.
    Let q = 7 is a prime.
    The compound proposition can be written as $\neg(p \vee q)$.

- **Converse:**

  The **converse** of $p \rightarrow q$ is $q \rightarrow p$.

  (the two do not always have the same truth value)

- **Contrapositive:**

  The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

  (the two always have the same truth value)

- Example:
  - Proposition:
    If it is raining, then there are clouds in the sky.

  - Converse:
    If there are clouds in the sky, then it is raining.

  - Contrapositive:
    If there are no clouds in the sky, then it is not raining.

- **Sufficient and Necessary Conditions:**

  If $p \rightarrow q$, then p is a **sufficient condition** for q, and q is a **necessary condition** for p.

  - To pass this course, it is necessary to work hard.
    $p \rightarrow q$; but, q is not sufficient for p.

  Note: p is called the **antecedent**, q is called the **consequent**.

Quantifiers

- $\forall$ - **Universal Quantifier - 'for all':**

  $\forall x\ p(x)$ - compound proposition

  $\forall x\ p(x)$ is true if $p(x)$ is true for every x in U; otherwise $\forall x\ p(x)$ is false.

- $\exists$ - **Existential Quantifier - 'there exists':**

  $\exists x\ p(x)$ - compound proposition

  $\exists x\ p(x)$ is true if $p(x)$ is true for at least one x in U; otherwise $\exists x\ p(x)$ is false.

- Example:

  - For each n in $\mathbb{N}$, let $p(n)$ be the proposition $n^2 = n$.

    $\forall n\ p(n)$ is false. (e.g., $n = 3$)

    $\exists n\ p(n)$ is true. ($n = 0$ or $n = 1$)

- Examples - Use of $\forall$ and $\exists$ as abbreviations:

  - $x + y = y + x$ for all $x,y \in \mathbb{R}$

    $x + y = y + x \ \forall x,y \in \mathbb{R}$

  - $2^n = n^2$ for some $n \in \mathbb{N}$

    $\exists n \in \mathbb{N}$ such that $2^n = n^2$.

- Examples - omission of understood quantifiers:

  $(x + y) + z = x + (y + z) \qquad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}$

  $(x + y) + z = x + (y + z) \qquad \forall x \forall y \forall z \in \mathbb{R}$

  $(x + y) + z = x + (y + z) \qquad \forall x,y,z \in \mathbb{R}$

  $(x + y) + z = x + (y + z)$ for all $x,y,z \in \mathbb{R}$

- Proving $\forall x\ p(x)$ is false:

  To **disprove** $\forall x\ p(x)$, it is enough to show that one of its propositions is false. This can be done with a **counterexample.**

- Example:
  - All prime numbers are odd.
    The number 2 is a counterexample.

Propositional Calculus

A formal set of rules for analyzing and manipulating propositions. It provides a mechanical method for calculating the truth value of complicated propositions.

- **Truth Tables:**

  Let:

  TRUE = T = 1
  FALSE = F = 0

- Truth Table for $\neg p$:

p	$\neg p$
0	1
1	0

- Truth Table for $p \vee q$:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

- Truth Table for $p \wedge q$:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

- Truth Table for $p \rightarrow q$:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Truth Tables for Compound Propositions

A truth table for a compound proposition built up from propositions p,q,r,... is a table giving the truth values of the compound proposition in terms of the truth values of p,q,r,.... We call p,q,r,... the **variables** of the truth table and of the compound proposition.

- Example: $(p \wedge q) \vee \neg(p \rightarrow q)$

p	q	$p \wedge q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$(p \wedge q) \vee \neg(p \rightarrow q)$
0	0	0	1	0	0
0	1	0	1	0	0
1	0	0	0	1	1
1	1	1	1	0	1

One can use a simpler truth table, with the same thought process by writing the truth values under the connectives.

	p	q	$(p \wedge q)$	$\vee$	$\neg$	$(p \rightarrow q)$
	0	0	0	**0**	0	1
	0	1	0	**0**	0	1
	1	0	0	**1**	1	0
	1	1	1	**1**	0	1
step	1	1	2	4	3	2

- Example: (p → q) ∧ [(q ∧ ¬r) → (p ∨ r)]

	p	q	r	(p → q)	∧	[(q	∧	¬	r)	→	(p	∨	r)]
	0	0	0	1	**1**		0	1		1		0	
	0	0	1	1	**1**		0	0		1		1	
	0	1	0	1	**0**		1	1		0		0	
	0	1	1	1	**1**		0	0		1		1	
	1	0	0	0	**0**		0	1		1		1	
	1	0	1	0	**0**		0	0		1		1	
	1	1	0	1	**1**		1	1		1		1	
	1	1	1	1	**1**		0	0		1		1	
step	1	1	1	2	5		3	2		4		2	

- Example: (p → q) ∧ (q → p)

	p	q	(p→q)	∧	(q→p)
	0	0	1	**1**	1
	0	1	1	**0**	0
	1	0	0	**0**	1
	1	1	1	**1**	1
#	1	1	2	3	2

- The **Biconditional** p ↔ q:

p	q	p ↔ q
0	0	1
0	1	0
1	0	0
1	1	1

Tautologies

A compound proposition that is always true is called a **tautology.**

- Example: p → p is a tautology

p	p → p
0	1
1	1

- Example: [p ∧ (p → q)] → q is a tautology

	p	q	[p	∧	(p	→	q)]	→	q
	0	0		0		1		**1**	
	0	1		0		1		**1**	
	1	0		0		0		**1**	
	1	1		1		1		**1**	
step	1	1		3		2		4	

- Example: ¬(p ∨ q) ↔ (¬p ∧ ¬q) is a tautology

	p	q	¬	(p	∨	q)	↔	(¬	p	∧	¬	q)
	0	0	1		0		**1**	1		1	1	
	0	1	0		1		**1**	1		0	0	
	1	0	0		1		**1**	0		0	1	
	1	1	0		1		**1**	0		0	0	
step	1	1	3		2		4	2		3	2	

Contradictions

A compound proposition that is always false is called a **contradiction.**

- Example: $p \land \neg p$ is a contradiction

p	$\neg p$	$p \land \neg p$
0	1	0
1	0	0

Logical Equivalence

Two compound propositions, P and Q, are **logically equivalent**, $P \Leftrightarrow Q$, if they have the same truth values for all possible choices of truth values of the variables; i.e.,

$P \Leftrightarrow Q$ iff $P \leftrightarrow Q$ is a tautology.

- **Logical Equivalences:**

1. $\neg\neg p \Leftrightarrow p$	double negation
2a. $(p \vee q) \Leftrightarrow (q \vee p)$	commutative laws
b. $(p \wedge q) \Leftrightarrow (q \wedge p)$	
c. $(p \leftrightarrow q) \Leftrightarrow (q \leftrightarrow p)$	
3a. $[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$	associative laws
b. $[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$	
4a. $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$	distributive laws
b. $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$	
5a. $(p \vee p) \Leftrightarrow p$	idempotent laws
b. $(p \wedge p) \Leftrightarrow p$	
6a. $(p \vee c) \Leftrightarrow p$	identity laws
b. $(p \vee t) \Leftrightarrow t$	(t - tautology)
c. $(p \wedge c) \Leftrightarrow c$	(c - contradiction)
d. $(p \wedge t) \Leftrightarrow p$	

7a. $(p \lor \neg p) \Leftrightarrow t$
b. $(p \land \neg p) \Leftrightarrow c$

8a. $\neg(p \lor q) \Leftrightarrow (\neg p \land \neg q)$ DeMorgan's laws
b. $\neg(p \land q) \Leftrightarrow (\neg p \lor \neg q)$
c. $(p \lor q) \Leftrightarrow \neg(\neg p \land \neg q)$
d. $(p \land q) \Leftrightarrow \neg(\neg p \lor \neg q)$

9. $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ contrapositive

10a. $(p \rightarrow q) \Leftrightarrow (\neg p \lor q)$ implication
b. $(p \rightarrow q) \Leftrightarrow \neg(p \land \neg q)$

11a. $(p \lor q) \Leftrightarrow (\neg p \rightarrow q)$
b. $(p \land q) \Leftrightarrow \neg(p \rightarrow \neg q)$

12a. $[(p \rightarrow r) \land (q \rightarrow r)] \Leftrightarrow [(p \lor q) \rightarrow r]$
b. $[(p \rightarrow q) \land (p \rightarrow r)] \Leftrightarrow [p \rightarrow (q \land r)]$

13. $(p \leftrightarrow q) \Leftrightarrow [(p \rightarrow q) \land (q \rightarrow p)]$ equivalence

14. $[(p \land q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$ exportation law

15. $(p \rightarrow q) \Leftrightarrow [(p \land \neg q) \rightarrow c]$ reductio ad absurdum

Logical Implications

Given two compound propositions P and Q, we say that P **logically implies** Q, $P \Rightarrow Q$, in case Q has truth value true whenever P has truth value true; i.e.,

$$P \Rightarrow Q \text{ iff } P \rightarrow Q \text{ is a tautology.}$$

- **Logical Implications:**

16. $p \Rightarrow (p \vee q)$	addition
17. $(p \wedge q) \Rightarrow p$	simplification
18. $(p \rightarrow c) \Rightarrow \neg p$	absurdity
19. $[p \wedge (p \rightarrow q)] \Rightarrow q$	modus ponens
20. $[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$	modus tollens
21. $[(p \vee q) \wedge \neg p] \Rightarrow q$	disjunctive syllogism
22. $p \Rightarrow [q \rightarrow (p \wedge q)]$	
23. $[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \Rightarrow (p \leftrightarrow r)$	transitivity of $\leftrightarrow$
24. $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$	transitivity of $\rightarrow$ or hypothetical syllogism

25a. $(p \rightarrow q) \Rightarrow [(p \vee r) \rightarrow (q \vee r)]$
b. $(p \rightarrow q) \Rightarrow [(p \wedge r) \rightarrow (q \wedge r)]$
c. $(p \rightarrow q) \Rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$

26a. $[(p \rightarrow q) \wedge (r \rightarrow s)] \Rightarrow [(p \vee r) \rightarrow (q \vee s)]$	constructive
b. $[(p \rightarrow q) \wedge (r \rightarrow s)] \Rightarrow [(p \wedge r) \rightarrow (q \wedge s)]$	dilemmas

27a. $[(p \rightarrow q) \wedge (r \rightarrow s)] \Rightarrow [(\neg q \vee \neg s) \rightarrow (\neg p \vee \neg r)]$	destructive
b. $[(p \rightarrow q) \wedge (r \rightarrow s)] \Rightarrow [(\neg q \wedge \neg s) \rightarrow (\neg p \wedge \neg r)]$	dilemmas

Methods of Proof

Given a set of hypotheses $H_1, H_2, \ldots, H_n$, we wish to infer the conclusion C.

- **Direct Proof:**

  Show $H_1 \wedge H_2 \wedge \cdots \wedge H_n \Rightarrow C$

- **Indirect Proof - Proof of the Contrapositive:**

  Show $\neg C \Rightarrow \neg(H_1 \wedge H_2 \wedge \cdots \wedge H_n)$

- Example:

  Given: $m, n \in \mathbb{N}$

  Prove:

  If $m + n \geq 73$, then $m \geq 37$ or $n \geq 37$.

  Contrapositive:

  If $m \leq 36$ and $n \leq 36$, then $m + n \leq 72$
  which is true by a general property of inequalities.
  ($a \leq c$ and $b \leq d$ implies $a + b \leq c + d$.)

- **Indirect Proof - Proof by Contradiction:**

  Show $H_1 \wedge H_2 \wedge \cdots \wedge H_n \wedge \neg C \Rightarrow$ a contradiction

- Example:

  Prove: the product of two odd integers is odd.

  Proof by contradiction:

  - Assume $m,n \in \mathbb{Z}$, but that $m \cdot n$ is even.
  - $\exists j,k \in \mathbb{Z}$ so that $m = 2j + 1$ and $n = 2k + 1$.
  - $m \cdot n = 4jk + 2j + 2k + 1$ = an odd number.
  - This is a contradiction.

  Direct proof:

  - Consider odd integers $m,n \in \mathbb{Z}$.
  - $\exists j,k \in \mathbb{Z}$ so that $m = 2j + 1$ and $n = 2k + 1$.
  - $m \cdot n = 4jk + 2j + 2k + 1$ = an odd number.

- **Proof by Cases:**

  An implication of the form $H_1 \lor H_2 \lor \cdots \lor H_n \Rightarrow C$ is equivalent to $H_1 \Rightarrow C$ and $H_2 \Rightarrow C$ and $\cdots$ and $H_n \Rightarrow C$ and can by proven by proving each $H_i \Rightarrow C$ for all i.

- Example:

  Prove: $\forall n \in \mathbb{N}$, $n^3 + n$ is even.

  Note: $n^3 + n = n(n^2 + 1)$

  Case 1: n is even

  - $n(n^2 + 1)$ must be even

  Case 2: n is odd

  - n^2 is odd; $n^2 + 1$ must be even; $n(n^2 + 1)$ is even

- **Vacuous Proof:**

  An implication $P \Rightarrow Q$ is said to be **vacuously true** or **true by default** if P is false.

- **Trivial Proof:**

  An implication $P \Rightarrow Q$ is said to be **trivially true** if Q is true.

- **Constructive and Nonconstructive Proofs:**

  Proofs for the existence of mathematical objects satisfying certain properties can be either **constructive** or **nonconstructive.**

  - **Constructive Proof:**

    Specifies the object or indicates how it can be determined by some explicit procedure or algorithm.

  - **Nonconstructive Proof:**

    Establishes the existence of objects by some indirect means, such as a proof by contradiction, without giving directions for how to find them.

More Propositional Calculus

- **Substitution Rule (a):**

  If a compound proposition P is a tautology and if all occurrences of some variable of P, say p, are replaced by the same proposition E, then the resulting compound proposition P^* is also a tautology.

- Example:

  Given:

  $P = [p \wedge (p \rightarrow q)] \rightarrow q$ (modus ponens - tautology)

  Let:

  $E = q \rightarrow r$

  Replacing p by E in P gives:

  $P^* = [(q \rightarrow r) \wedge ((q \rightarrow r) \rightarrow q)] \rightarrow q$ which must be a tautology

q	r	[(q	→	r)	∧	((q	→	r)	→	q)]	→	q
0	0		1		0		1		0		**1**	
0	1		1		0		1		0		**1**	
1	0		0		0		0		1		**1**	
1	1		1		1		1		1		**1**	

  Replacing q by E in P gives:

  $P^{**} = [p \wedge (p \rightarrow (q \rightarrow r))] \rightarrow (q \rightarrow r)$ which is also a tautology

- **Substitution Rule (b):**

  If a compound proposition P contains a proposition Q and if Q is replaced by a logically equivalent proposition Q^*, then the resulting compound proposition P^* is logically equivalent to P.

- Example:

  Given:

  $P = \neg[(p \rightarrow q) \wedge (p \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$ (not a tautology)

  Let:

  $Q = p \rightarrow q$ and $Q^* = (\neg p \vee q)$ (logically equivalent)

  Replacing Q by Q^* in P gives:

  $\neg[(\neg p \vee q) \wedge (p \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$ (logically equiv. to P)

- Example - Using Rules (a) and (b) to build equivalence chains:

Equivalence Proposition	Explanations
$(p \vee q) \vee (p \vee r)$	Given
$[(p \vee q) \vee p] \vee r$	Associative Law 3a
$[p \vee (q \vee p)] \vee r$	Associative Law 3a
$[p \vee (p \vee q)] \vee r$	Commutative Law 2a
$[(p \vee p) \vee q] \vee r$	Associative Law 3a
$[p \vee q] \vee r$	Idempotent Law 5a

Formal Proofs

- Given a set of **hypotheses** and some **conclusion** C, a **formal proof** of C from the hypotheses consists of a **chain of propositions** $P_1,P_2,...,P_n,C$, ending with C in which each P_i is either:
  - a hypothesis or
  - a tautology or
  - a consequence of previous members of the chain by using the allowable rules of inference.

- A **theorem** is a statement of the form 'If H then C', where H is a set of hypotheses and C is a conclusion.

- **Rules of Inference:**

  - Substitution Rules (a) and (b)
  - Rules based on logical implications of the form:

    $P_1 \wedge P_2 \wedge \cdots \wedge P_m \Rightarrow Q.$

  (If $P_1,P_2,\cdots,P_m$ have already appeared in the chain and $P_1 \wedge P_2 \wedge \cdots \wedge P_m \Rightarrow Q$ is true, then we are allowed to add Q to the chain.)

- Example:

  Given:

  $(B \vee S) \rightarrow P$

  $B \wedge S$

  Conclusion:

  P

<u>Proof</u>	<u>Reason</u>
1. $(B \vee S) \rightarrow P$	hypothesis
2. $B \wedge S$	hypothesis
3. B	2; simplification rule 17
4. $B \vee S$	3; addition rule 16
5. P	1,4; modus ponens rule 19

Rules of Inference

28. **addition**

$$\frac{P}{\therefore P \lor Q}$$

29. **simplification**

$$\frac{P \land Q}{\therefore P}$$

30. **modus ponens**

$$\frac{\begin{array}{l}P\\ P \to Q\end{array}}{\therefore Q}$$

31. **modus tollens**

$$\frac{\begin{array}{l}P \to Q\\ \neg Q\end{array}}{\therefore \neg P}$$

32. **disjunctive syllogism**

$$\frac{\begin{array}{l}P \lor Q\\ \neg P\end{array}}{\therefore Q}$$

33. **hypothetical syllogism**

$$\frac{\begin{array}{l}P \to Q\\ Q \to R\end{array}}{\therefore P \to R}$$

34. **conjunction**

$$\frac{\begin{array}{l}P\\ Q\end{array}}{\therefore P \land Q}$$

- Example:

Conclude $s \to r$ from $p \to (q \to r)$; $p \vee \neg s$; and q.

1. $p \to (q \to r)$	hypothesis
2. $p \vee \neg s$	hypothesis
3. q	hypothesis
4. $\neg s \vee p$	2; commutative law 2a
5. $s \to p$	4; implication rule 10a
6. $s \to (q \to r)$	1,5; hypothetical syllogism rule 33
7. $(s \wedge q) \to r$	6; exportation rule 14
8. $(q \wedge s) \to r$	7; commutative law 2b
9. $q \to (s \to r)$	8; exportation rule 14
10. $s \to r$	3,9; modus ponens rule 30

- Example:

Conclude $\neg p$ by contradiction from $p \rightarrow (q \wedge r)$; $r \rightarrow s$; and $\neg(q \wedge s)$.

1. $p \rightarrow (q \wedge r)$	hypothesis
2. $r \rightarrow s$	hypothesis
3. $\neg(q \wedge s)$	hypothesis
4. $\neg(\neg p)$	negation of the conclusion
5. p	4; double negation rule 1
6. $q \wedge r$	1,5; modus ponens rule 30
7. q	6; simplification rule 29
8. $r \wedge q$	6; commutative law 2b
9. r	8; simplification rule 29
10. s	2,9; modus ponens rule 30
11. $q \wedge s$	7,10; conjunction rule 34
12. $(q \wedge s) \wedge \neg(q \wedge s)$	3,11; conjunction rule 34
13. contradiction	12; rule 7b

- Direct Proof:

4. $\neg(s \wedge q)$	3; commutative law 2b
5. $s \rightarrow \neg q$	4; implication 10b
6. $r \rightarrow \neg q$	2,5; hypothetical syllogism rule 33
7. $\neg(r \wedge q)$	6; implication 10b
8. $\neg(q \wedge r)$	7; commutative law 2b
9. $\neg p$	1,8; modus tollens rule 31

Analysis of Arguments

- **Valid Inferences and Valid Proofs:**

  Formal rules of inference are sometimes called **valid inferences**, and formal proofs are called **valid proofs.**

- **Fallacies:**

  A sequence of propositions that fails to meet the requirements for being a formal proof is called a **fallacy**.

- Example - analyze the following:

  "If I study or if I am a genius, then I will pass the course. If I pass the course, then I will be allowed to take the next course. Therefore, if I am not allowed to take the next course, then I am not a genius."

  Let:

  s = I study
  g = I am a genius
  p = I will pass the course
  n = I will be allowed to take the next course

  Hypotheses:

  $(s \lor g) \rightarrow p$
  $p \rightarrow n$

  Conclusion:

  $\neg n \rightarrow \neg g$

<u>Proof</u>	<u>Explanations</u>
1. $(s \lor g) \rightarrow p$	hypothesis
2. $p \rightarrow n$	hypothesis
3. $g \rightarrow (g \lor s)$	addition (rule 16) - tautology
4. $g \rightarrow (s \lor g)$	3; commutative law 2a
5. $g \rightarrow p$	4,1; hypothetical syllogism (rule 33)
6. $g \rightarrow n$	5,2; hypothetical syllogism (rule 33)
7. $\neg n \rightarrow \neg g$	6; contrapositive (rule 9)

- Alternate Proof:

Hypotheses:

$(s \vee g) \rightarrow p$

$p \rightarrow n$

Conclusion:

$\neg n \rightarrow \neg g$

Proof	Explanations
1. $(s \vee g) \rightarrow p$	hypothesis
2. $p \rightarrow n$	hypothesis
3. $(s \vee g) \rightarrow n$	1,2; hypothetical syllogism (rule 33)
4. $(s \rightarrow n) \wedge (g \rightarrow n)$	3; implication (rule 12a)
5. $g \rightarrow n$	4; simplification (rule 29)
6. $\neg n \rightarrow \neg g$	5; contrapositive (rule 9)

- Example - analyze the following:

"If I study or if I am a genius, then I will pass the course. I will not be allowed to take the next course. If I pass the course, then I will be allowed to take the next course. Therefore, I did not study."

Let:

s = I study
g = I am a genius
p = I will pass the course
n = I will be allowed to take the next course

Hypotheses:

$(s \vee g) \rightarrow p$
$\neg n$
$p \rightarrow n$

Conclusion:

$\neg s$

Proof	Explanations
1. $(s \vee g) \rightarrow p$	hypothesis
2. $s \rightarrow (s \vee g)$	addition (rule 16) - tautology
3. $s \rightarrow p$	2,1; hypothetical syllogism (rule 33)
4. $p \rightarrow n$	hypothesis
5. $s \rightarrow n$	3,4; hypothetical syllogism (rule 33)
6. $\neg n$	hypothesis
7. $\neg s$	5,6; modus tollens (rule 31)

- Alternate Proof:

Hypotheses:

$(s \vee g) \rightarrow p$

$\neg n$

$p \rightarrow n$

Conclusion:

$\neg s$

Proof	Explanations
1. $(s \vee g) \rightarrow p$	hypothesis
2. $\neg n$	hypothesis
3. $p \rightarrow n$	hypothesis
4. $(s \vee g) \rightarrow n$	1,3; hypothetical syllogism (rule 33)
5. $\neg(s \vee g)$	2,4; modus tollens (rule 31)
6. $\neg s \wedge \neg g$	5; DeMorgan's law
7. $\neg s$	6; simplification

- Example - analyze the following:

  We are given that if a program does not fail, then it begins and terminates. We know that our program began and failed. We conclude that our program did not terminate.

  Let:

  B = the program begins
  T = the program terminates
  F = the program fails

  Hypotheses:

  $\neg F \rightarrow (B \wedge T)$
  $B \wedge F$

  Conclusion:

  $\neg T$

Proof	Explanations
1. $\neg F \rightarrow (B \wedge T)$	hypothesis
2. $B \wedge F$	hypothesis
3. $(\neg F \rightarrow B) \wedge (\neg F \rightarrow T)$	1; implication (rule 12b)
4. $\neg F \rightarrow T$	3; simplification (rule 29)
5. F	2; simplification (rule 29)
6. $\neg T$	4,5; ???

- The above proof is a fallacy!

- Note: $[[\neg F \rightarrow (B \wedge T)] \wedge (B \wedge F)] \rightarrow \neg T$ is not a tautology.