Probability

- Our setting for probability is a sample space Ω and a probability P: for events $E \subseteq \Omega$, P(E) is a number [0,1] and represents the probability of event E.

- **Conditional Probability:**

  Given a sample space Ω and events E and S. With P(S) > 0, the conditional probability P(E|S), read "the probability of E given S" is defined as

$$P(E|S) = \frac{P(E \cap S)}{P(S)}$$

- Example:

  A black die and a red die are both tossed.

  Ω is the set of outcomes.

  Events:

  B = "value on black die $\leq$ 3"

  R = "value on red die is $\geq$ 5"

  S = "the sum of the values is $\geq$ 8"

  Probabilities:

  P(B) = 18/36 = 1/2; P(R) = 12/36 = 1/3

  P(S) = 15/36 = 5/12

  $P(R \cap S)$ = 9/36 = 1/4; $P(B \cap S)$ = 3/36 = 1/12

$$P(R|S) = \frac{P(R \cap S)}{P(S)} = \frac{9/36}{15/36} = \frac{9}{15} = \frac{3}{5}$$

$$P(B|S) = \frac{P(B \cap S)}{P(S)} = \frac{3/36}{15/36} = \frac{3}{15} = \frac{1}{5}$$

- **Independent Events:**

  If events A and B are **independent**, then

  $P(A \cap B) = P(A) \cdot P(B)$.

- For independent events A and B,

  $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

- Example - return to previous two-dice problem:

  $P(B \cap R) = 6/36 = 1/6 = P(B) \cdot P(R)$

  $$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{6/36}{12/36} = \frac{6}{12} = \frac{1}{2} = P(B)$$

  $$P(R|B) = \frac{P(R \cap B)}{P(B)} = \frac{6/36}{18/36} = \frac{6}{18} = \frac{1}{3} = P(R)$$

  Note: Events B and R are independent.

  $$P(S|R) = \frac{P(S \cap R)}{P(R)} = \frac{9/36}{12/36} = \frac{9}{12} = \frac{3}{4} \neq P(S) = \frac{5}{12}$$

  Note: Events S and R are not independent.

- **General Statement of Independence:**

  Events $A_1, A_2, \ldots, A_n$ are **independent** if

  $$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

  for all nonempty subsets of $J = \{1,2,\ldots,n\}$.

- Given events A and B:

  $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$.

- Example:

  Two cards are drawn from a deck of cards.

  - What is the probability that both cards are aces if the first card is put back, and the deck shuffled, before the second card is drawn?
  - Answer:

    Events:

    A = "the first card is an ace"

    B = "the second card is an ace"

    $P(A \cap B) = P(A)P(B|A) = (1/13)(1/13) = 1/169$.
  - What is the probability that both cards are aces if the first card is not put back?

    $P(A \cap B) = P(A)P(B|A) = (4/52)(3/51) = 1/221$.

Total Probability

- **Law of Total Probability**:

  If events $A_1, A_2, \ldots, A_k$ partition Ω and $P(A_i) > 0$ for each i, then for each event B we have

  $$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_k)P(B|A_k)$$
  $$= \sum_{i=1}^{k} P(A_i) \cdot P(B|A_i)$$

- Example:

  A company purchases a certain part from three firms and keeps a record of how many are defective. A part is selected at random.

Firm →	A	B	C
Fraction of parts purchased	.50	.20	.30
Fraction of defective parts	.01	.04	.02

Events:

A = "the part came from A"
B = "the part came from B"
C = "the part came from C"
D = "the part is defective"

$$P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)$$
$$= (.50)(.01) + (.20)(.04) + (.30)(.02) = .019$$

Bayes' Formula

- **Bayes Formula:**

  Suppose that the events $A_1, A_2, \ldots, A_k$ partition Ω and $P(A_i) > 0$ for each i, and suppose that B is any event for which $P(B) > 0$. Then,

  $$P(A_j|B) = \frac{P(A_j) \cdot P(B|A_j)}{P(B)}$$

  for each j, where

  $$P(B) = \sum_{i=1}^{k} P(A_i) \cdot P(B|A_i)$$

- Example:

  For the randomly selected part of the previous example:

  $$P(A|D) = \frac{P(A) \cdot P(D|A)}{P(D)} = \frac{(.50)(.01)}{.019} \approx .263$$

  $$P(B|D) = \frac{P(B) \cdot P(D|B)}{P(D)} = \frac{(.20)(.04)}{.019} \approx .421$$

  $$P(C|D) = \frac{P(C) \cdot P(D|C)}{P(D)} = \frac{(.30)(.02)}{.019} \approx .316$$

- **General Formula for Joint Probability:**

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \middle| \bigcap_{j=1}^{i-1} A_j\right)$$

- Example:

  Four cards are drawn from a deck without replacement. What is the probability that the first two drawn will be aces and the second two drawn will be kings?

  $P(A_1 \cap A_2 \cap K_3 \cap K_4)$
  $= P(A_1) \cdot P(A_2 | A_1) \cdot P(K_3 | A_1 \cap A_2) \cdot P(K_4 | A_1 \cap A_2 \cap K_3)$
  $= (4/52)(3/51)(4/50)(3/49) \approx .000022$

Random Variables

- A function from Ω to $\mathbb{R}$ is called a **random variable**.

- Given a random variable X on Ω and a condition C:
  $\{X \text{ satisfies } C\} = \{\omega \in \Omega : X(\omega) \text{ satisfies } C\}$

- Examples:
  $\{5 < X \leq 7\} = \{\omega \in \Omega : 5 < X(\omega) \leq 7\}$
  $\{X = a \text{ or } X^2 = b\} = \{\omega \in \Omega : X(\omega) = a \text{ or } X(\omega)^2 = b\}$

- When writing probabilities, we usually drop the braces:
  $P(X = 2) = P(\{X = 2\}) = P(\{\omega \in \Omega : X(\omega) = 2\}$
  $P(X \in A) = P(\{\omega \in \Omega : X(\omega) \in A\}$ for $A \subseteq \mathbb{R}$

- $X(\Omega) = \{X(\omega) : \omega \in \Omega\}$ is the **value set** of X.

- Example:
  A black die and a red die are both tossed.
  Random Variables:
  - X_B is the value on the black die,
  - X_R is the value on the red die,
  - X_S is the sum of X_B and X_R

  Probabilities:
  - $P(X_B \leq 3) = 18/36 = 1/2$; $P(X_R \geq 5) = 12/36 = 1/3$
  - $P(X_S \geq 8) = 15/36 = 5/12$
  - $P(X_B \leq 3 \mid X_S \geq 8) = 3/15$
  - $P(X_R \geq 5 \mid X_S \geq 8) = 9/15$

Independent Random Variables

- **Independence of Two Random Variables**:
  Two random variables X and Y on Ω are **independent** if
  $P(X \in I \text{ and } Y \in J) = P(X \in I) \cdot P(Y \in J)$
  for all intervals I and J in $\mathbb{R}$.

- **General Independence of Random Variables**:
  A sequence $X_1, X_2, \ldots, X_n$ of random variables on Ω is **independent** if

$$P(X_i \in J_i \; for \; i = 1, 2, \ldots, n) = \prod_{i=1}^{n} P(X_i \in J_i)$$

  for all intervals $J_1, J_2, \ldots, J_n$ in $\mathbb{R}$.

- Example:
  - A fair coin is tossed n times. Let $X_i = 1$ if the ith toss is a head, and 0 otherwise.

$$P(X_i = x_i \; for \; i = 1, 2, \ldots, n) = \prod_{i=1}^{n} P(X_i = x_i) = \frac{1}{2^n}$$

  - Let $S_n = X_1 + X_2 + \cdots + X_n$.
    The value set of $S_n = \{0, 1, 2, \ldots, n\}$

$$P(S_n = k) = \frac{1}{2^n}\binom{n}{k} \qquad \text{for } k \in \{0, 1, 2, \ldots, n\}$$

- **Discrete Random Variables:**
  A random variable is discrete if its value set is finite or an infinite sequence.

- **Probability Density Function - pdf:**
  The pdf of X is the function f_X defined on $\mathbb{R}$ by
  $f_X(x) = f(x) = P(X = x)$ for $x \in \mathbb{R}$
  (The pdf is sometimes called the **probability distribution.**)

- Example:
  A black die and a red die are both tossed.

  Random Variables:
  X_B is the value on the black die,
  X_R is the value on the red die,
  X_S is the sum of X_B and X_R

  Probability Density Functions:

$$f_{X_B}(x) = f_{X_R}(x) = \begin{cases} 1/6 & if\ x = 1,2,3,4,5, or\ 6 \\ 0 & for\ other\ x \end{cases}$$

$$f_{X_S}(x) = \begin{cases} 1/36 & if\ x = 2\ or\ 12 \\ 2/36 & if\ x = 3\ or\ 11 \\ 3/36 & if\ x = 4\ or\ 10 \\ 4/36 & if\ x = 5\ or\ 9 \\ 5/36 & if\ x = 6\ or\ 8 \\ 6/36 & if\ x = 7 \\ 0 & for\ other\ x \end{cases}$$

- **Cumulative Distribution Function - cdf:**

  The cdf of X is the function F_X, defined on $\mathbb{R}$ by

  $F_X(y) = F(y) = P(X \le y)$ for $y \in \mathbb{R}$.

- For a discrete random variable:

  $$F_X(y) = \sum_{x \le y} f_X(x)$$

- Example:

  A black die and a red die are both tossed.

  Random Variables:

  X_B is the value on the black die,
  X_R is the value on the red die,
  X_S is the sum of X_B and X_R

  Cumulative Distribution Functions:

$$F_{X_B}(y) = F_{X_R}(y) = \begin{cases} 0 & for\ y < 1 \\ 1/6 & for\ 1 \le y < 2 \\ 2/6 & for\ 2 \le y < 3 \\ 3/6 & for\ 3 \le y < 4 \\ 4/6 & for\ 4 \le y < 5 \\ 5/6 & for\ 5 \le y < 6 \\ 1 & for\ y \ge 6 \end{cases}$$

$$F_{X_S}(y) = \sum_{x \le y} f_{X_S}(x)$$

- Example:
  - A fair coin is tossed n times. Let $X_i = 1$ if the ith toss is a head, and 0 otherwise.

    $P(X_i = 0) = P(X_i = 1) = 1/2$

    $$F_{X_i}(y) = \begin{cases} 0 & \textit{for } y < 0 \\ 1/2 & \textit{for } 0 \le y < 1 \\ 1 & \textit{for } y \ge 1 \end{cases}$$

  - Let $S_n = X_1 + X_2 + \cdots + X_n$.

    The value set of $S_n = \{0,1,2,...,n\}$

    $P(S_n = k) = \frac{1}{2^n}\binom{n}{k}$ for $k \in \{0,1,2,...,n\}$

    $$F_{S_n}(y) = \sum_{x \le y} \frac{1}{2^n}\binom{n}{x}$$

- For discrete random variables, if the value set can be written $\{x_1,x_2,...\}$ where $x_1 < x_2 < x_3 < \cdots$, then

  $f_X(x_1) = F_X(x_1)$,

  $f_X(x_k) = F_X(x_k) - F_X(x_{k-1})$, for $k \ge 2$,

  $f_X(x) = 0$ for other x.

- **Equivalence of Random Variables:**

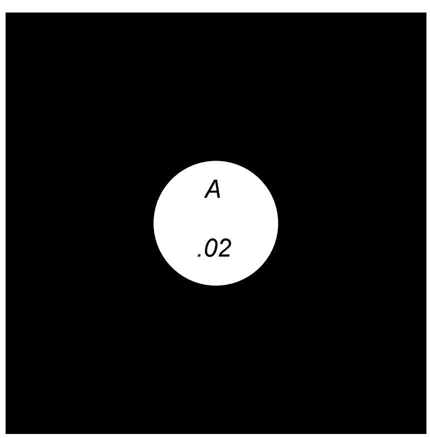
  Two random variables are equivalent if they are **identically distributed** (i.e., they have the same distributions).

Expectation and Standard Deviation

- **Expected Value of a Random Variable:**
  Given a random variable X on a finite sample space Ω, its **expected value,** (**expectation** or **mean**) is defined by:
  $$E(X) = \mu = \sum_{\omega \in \Omega} X(\omega)\cdot P(\{\omega\})$$

- Example:
  A fair die is tossed.
  Random variable X is the value on the die.
  $\mu = E(X) = 1/6(1 + 2 + 3 + 4 + 5 + 6) = 3.5$

- Example:
  A dart player knows from experience that each time he throws a dart at the dart board shown below, he will hit the various regions with the probabilities indicated. A concessionaire owns the dart board and offers to pay $10 whenever the dart hits region A, $3 whenever it hits region B, and $1 whenever it hits region C. What is the player's average income per throw of the dart?
  $\mu = 10(.02) + 3(.06) + 1(.10) + 0(.82) = \$0.48.$



- **Theorem:**
  Let X and Y be random variables on a finite sample space Ω
  (a) $E(X + Y) = E(X) + E(Y)$.
  (b) $E(aX) = a \cdot E(X)$ for $a \in \mathbb{R}$.
  (c) $E(a) = a$ for $a \in \mathbb{R}$.

- **Corollary:**
  For any random variable X,
  $E(X - \mu) = 0$.

- **Theorem:**
  For a random variable on a finite sample space Ω,
  $$E(X) = \sum_x x \cdot P(X = x) = \sum_x x \cdot f_X(x).$$

- Example:
  A black die and a red die are both tossed.

  Random Variables:
  X_B is the value on the black die,
  X_R is the value on the red die,
  X_S is the sum of X_B and X_R

  $$E(X_S) = \sum_{(k,l) \in \Omega} X_S(k,l) \cdot P(k,l) = \frac{1}{36}(2 + 3 + 3 + 4 + \cdots + 11 + 12) = 7$$
  or
  $$E(X_S) = \sum_x x \cdot P(X_S = x) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 12 \cdot \frac{1}{36} = 7$$
  or
  $E(X_S) = E(X_B + X_R) = E(X_B) + E(X_R) = 3.5 + 3.5 = 7$

- Example

  A fair coin is tossed n times. Let $X_i = 1$ if the ith toss is a head, and 0 otherwise. Let $S_n = X_1 + X_2 + \cdots + X_n$.

  - What is the expected number of heads?

  $$E(S_n) = \sum_{k=0}^{n} k \cdot P(S_n = k) = \sum_{k=0}^{n} k \cdot \frac{1}{2^n} \binom{n}{k}$$

  - Note:

  $E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = 1/2.$

  $$E(S_n) = \sum_{i=1}^{n} E(X_i) = \frac{n}{2}$$

  - We have just proven:

  $$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$$

- **Lemma:**

  Let Z be a random variable on Ω. Suppose that we are given a partition of Ω such that Z is constant on each set of the partition. Then, E(Z) is he sum of the products of the probability of each set times the [constant] value that Z takes on the set.

- **Theorem:**

  For a random variable X with a finite value set and a function $\phi: \mathbb{R} \to \mathbb{R}$, we have

  $$E(\phi \circ X) = \sum_x \phi(x) \cdot P(X = x)$$

- **Corollary:**

  For X as above,

  $$E(X^2) = \sum_x x^2 \cdot P(X = x)$$

  $$E((X - \mu)^2) = \sum_x (x - \mu)^2 \cdot P(X = x)$$

- **Theorem:**

  For random variables X and Y with finite value sets and a function $\psi: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, such that $\psi(X,Y)(\omega) = \psi(X(\omega),Y(\omega))$,

  $$E(\psi(X,Y)) = \sum_x \sum_y \psi(x,y) \cdot P(X = x \text{ and } Y = y)$$

- **Corollary:**

  $$E(XY) = \sum_x \sum_y xy \cdot P(X = x \text{ and } Y = y)$$

- **Theorem:**

  For independent random variables X and Y,

  $$E(XY) = E(X) \cdot E(Y)$$

Variance and Standard Deviation

- **Variance of a Random Variable - σ^2:**

  The **variance** of a random variable X, with mean μ, is

  $$V(X) = \sigma^2_X = \sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$$

- **The Standard Deviation of a Random Variable - σ:**

  The **standard deviation** σ of a random variable X is the square root of the variance.

- Example:

  Let X be the random variable that records the value of a tossed fair die.

  $$\mu = 3.5$$

  $$V(X) = \sigma^2 = \sum_{k=1}^{6} (k - \mu)^2 \cdot \frac{1}{6} = \frac{35}{12}$$

  $$\sigma = \sqrt{\frac{35}{12}} \approx 1.71$$

- Example:

  Let X be the coin tossing random variable so that P(X = 0) = P(X = 1) = 1/2. then

  $$\mu = 1/2$$

  $$V(X) = \left(0 - \frac{1}{2}\right)^2 \cdot \frac{1}{2} + \left(1 - \frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{4}$$

  $$\sigma = 1/2$$

- **Theorem:**

  For independent random variables $X_1, X_2, \ldots, X_n$

  $$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$$

- Example:

  A black die and a red die are both tossed.

  Random Variables:

  X_B is the value on the black die,

  X_R is the value on the red die,

  X_S is the sum of X_B and X_R

  From above:

  $\mu_B = \mu_R = 3.5$

  $V(X_B) = V(X_R) = 35/12$

  Calculate:

  $V(X_S) = V(X_B + X_R) = V(X_B) + V(X_R) = 35/6$

- Example:

  A fair coin is tossed n times. Let $X_i = 1$ if the ith toss is a head, and 0 otherwise. Let $S_n = X_1 + X_2 + \cdots + X_n$.

  From above:

  $V(X_i) = 1/4$

  Calculate:

  $$V(S_n) = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} \frac{1}{4} = \frac{n}{4}$$