Homework Solutions - Section 1.5

1.
f: $\mathbb{N} \to \mathbb{N}$; g: $\mathbb{N} \to \mathbb{Z}$
$f(n) = n^2 + 3$; $g(n) = 5n - 11$
(a) $f(1) = 1 + 3 = 4$; $g(1) = 5 - 11 = -6$
(b) $f(2) = 4 + 3 = 7$; $g(2) = 10 - 11 = -1$
(c) $f(3) = 9 + 3 = 12$; $g(3) = 15 - 11 = 4$
(d) $f(4) = 16 + 3 = 19$; $g(4) = 20 - 11 = 9$
(e) $f(5) = 25 + 3 = 28$; $g(5) = 25 - 11 = 14$
(f) $f(n) + g(n) = n^2 + 5n - 8 = n(n + 5) - 8$

This is even for all $n \in \mathbb{N}$ since $n(n + 5)$ is always the product of one even and one odd term, which is always even.

3.
$\Sigma = \{a,b\}$
(a) length(bab) = 3
(b) length(aaaaaaaa) = 8
(c) length(λ) = 0
(d) Im(length) = $\mathbb{N}$

5.
$f(m,n) = \lfloor n/2 \rfloor - \lfloor (m-1)/2 \rfloor$
(a) $f(0,0) = f(8,8) = f(-8,-8) = 1$; $f(73,73) = f(-73,-73) = 0$
(b) f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$f(n,n) = 1$ for n even
$f(n,n) = 0$ for n odd

7.
f: $\mathbb{R} \to \mathbb{R}$
$f(x) = x^3$ for $x \geq 1$
$f(x) = x$ for $0 \leq x < 1$
$f(x) = -x^3$ for $x < 0$
(a) $f(3) = 27$; $f(⅓) = ⅓$; $f(-⅓) = 1/27$; $f(-3) = 27$
(c) $\mathrm{Im}(f) = [0,\infty)$

13.
f: $\mathbb{R} \to \mathbb{R}$ with $f(x) = x^3 - 4x$
g: $\mathbb{R} \to \mathbb{R}$ with $g(x) = 1/(x^2 + 1)$
h: $\mathbb{R} \to \mathbb{R}$ with $h(x) = x^4$
(a) $f \circ f = f(f(x)) = f(x^3 - 4x) = (x^3 - 4x)^3 - 4(x^3 - 4x)$
(b) $g \circ g = g(g(x)) = g(1/(x^2 + 1)) = 1/((1/(x^2 + 1))^2 + 1)$
$= 1/(1/(x^2 + 1)^2 + 1) = (x^2 + 1)^2 / (1 + (x^2 + 1)^2)$
(c) $h \circ g = h(g(x)) = h(1/(x^2 + 1)) = (x^2 + 1)^{-4}$
(d) $g \circ h = g(h(x)) = g(x^4) = 1/(x^8 + 1) = (x^8 + 1)^{-1}$
(e) $f \circ g \circ h = f(g(h(x))) = f(1/(x^8 + 1)) = (x^8 + 1)^{-3} - 4(x^8 + 1)^{-1}$
(f) $f \circ h \circ g = f(h(g(x))) = f((x^2 + 1)^{-4}) = (x^2 + 1)^{-12} - 4(x^2 + 1)^{-4}$
(g) $h \circ g \circ f = h(g(f(x))) = h(g(x^3 - 4x)) = ((x^3 - 4x)^2 + 1)^{-4}$