Homework Solutions - Section 1.7

1.
$S = \{1,2,3,4,5\}$ $T = \{a,b,c,d\}$
(a) No - S is larger than T
(b) Yes - e.g., f(a)=1, f(b)=2, f(c)=3, f(d)=4
(c) Yes - e.g., f(1)=a, f(2)=b, f(3)=c, f(4)=f(5)=d
(d) No - T is smaller than S
(e) No - S and T are of different sizes

3.
$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $f(m,n) = 2^m 3^n$
(a) $f(2,1) = 2^2 3^1 = (4)(3) = 12$, $f(1,2) = 2^1 3^2 = (2)(9) = 18$, ...
(b) Let $f(m,n) = f(p,q)$, then $2^m 3^n = 2^p 3^q$.
    If $m < p$, then, if we divide both sides by 2^m, we get $3^n = 2^{p-m} 3^q$.
    Now 3^n and 3^p are both odd and if $m \neq p$, 2^{p-m} is even.
    Hence, we would have an odd number equal to an even number.
    This is a contradiction.
    Therefore, $f(m,n) = f(p,q)$, only if $m = n$ and $p = q$.
    Consequently, we have a one-to-one function.
(c) No, e.g., $f^{\leftarrow}(0) = \varnothing$, or $f^{\leftarrow}(5) = \varnothing$.
(d) $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $g(m,n) = 2^m 4^n$
    $g(2,0) = g(0,1) = 4$.
    Hence, g is not one-to-one.
    Note: $g(m,n) = 2^m 4^n = 2^m 2^{2n} = 2^{m+2n}$
    and $g(p,q) = 2^p 4^q = 2^{p+2q}$
    Hence, $g(m,n) = g(p,q)$ if $m+2n = p+2q$.

5.
f: $\mathbb{N}\to\mathbb{N}$; g: $\mathbb{N}\to\mathbb{N}$
f(n) = n + 1; g(n) = max{0, n - 1}
(a) f(0)=1, f(1)=2, f(2)=3, f(3)=4, f(4)=5, f(73)=74
(b) g(0)=0, g(1)=0, g(2)=1, g(3)=2, g(4)=3, g(73)=72
(c) If f(n) = f(p), then n+1=p+1 which implies n=p.
Therefore, f is one-to-one.
f is not onto since $f^{\leftarrow}(0) = \varnothing$.
(d) g is onto since $g^{\leftarrow}(n)$ contains n+1 for all $n \in \mathbb{N}$.
g is not one-to-one since $g^{\leftarrow}(0) = \{0,1\}$.
(e) $g \circ f = g(f(n)) = g(n+1) = \max\{0, n+1-1\} = \max\{0, n\} = n$
$f \circ g = f(g(0)) = f(0) = 1$

7.
(a) $f^{-1}(y) = (y - 3) / 2$
(b) $g^{-1}(y) = (y + 2)^{1/3}$
(c) $h^{-1}(y) = y^{1/3} + 2$
(d) $k^{-1}(y) = (y - 7)^3$

11.
All functions: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
SUM(m,n) = m + n; PROD(m,n) = mn
MAX(m,n) = max{m,n}; MIN(m,n) = min{m,n}
(a) All of them are onto
(b) None are one-to-one
SUM(2,1) = SUM(1,2)
PROD(2,1) = PROD(1,2)
MAX(2,1) = MAX(1,2)
MIN(2,1) = MIN(1,2)
(c) $SUM^{\leftarrow}(4) = \{(0,4),(1,3),(2,2),(3,1),(4,0)\}$
$PROD^{\leftarrow}(4) = \{(1,4),(2,2),(4,1)\}$
$MAX^{\leftarrow}(4) = \{(0,4),(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1),(4,0)\}$
$MIN^{\leftarrow}(4) = \{(4,n) : n \geq 4\} \cup \{(m,4) : m \geq 4\}$ (infinite number)