Homework Solutions - Section 2.3

5.
For $n \in \mathbb{P}$, we can write $n = 3k + r$, where $k \in \mathbb{N}$ and $r = 0,1$, or 2.
Therefore, $n^2 - 2 = (3k + r)^2 - 2 = 9k^2 + 6kr + r^2 - 2$
Now, $9k^2 + 6kr$ is divisible by 3.
For $r = 0,1$, or 2, $r^2 - 2$ is $-2,-1$, or 2 none of which are divisible by 3.
Hence, $n^2 - 2$ is not divisible by 3 for $n \in \mathbb{P}$.

7.
(a)
For $n \in \mathbb{P}$, we can write $n = 7k + r$, where $k \in \mathbb{N}$ and $r = 0,1,2,3,4,5,$or 6.
Therefore, $n^2 - 2 = (7k + r)^2 - 2 = 49k^2 + 14kr + r^2 - 2$
Now, $49k^2 + 14kr$ is divisible by 7.
For $r = 3$, $r^2 - 2 = 7$ which is divisible by 7. (Same is true for $r = 4$)
Hence, $n^2 - 2$ is divisible by 3 for some $n \in \mathbb{P}$.
(b) $n = 3$ or 4 or 10 or 11 or ... are counterexamples.
(c) 3, 11, 17, ...

9.
Let m and n be the two even integers.
We can write, $m = 2j$ and $n = 2k$ where $j,k \in \mathbb{Z}$
Therefore, $m \bullet n = 2j \bullet 2k = 4jk$ which a multiple of 4.

13.
(a) For $n \in \mathbb{N}$,
$n^4 - n^2 = n^2(n^2 - 1) = (n - 1)n^2(n + 1)$
This is the product involving three consecutive integers, one of which must be divisible by 3. Hence, $n^4 - n^2$ must be divisible by 3.
(b) From (a), at least one of the three consecutive integers must be even. Hence, $n^4 - n^2$ must be even.
(c) From (a) and (b), $n^4 - n^2$ must be an even multiple of 3, which makes it divisible by 6.