Homework Solutions - Section 3.4

1.

(a) $\parallel$ is an equivalence relation

R: $L_j \parallel L_j$ for all j (for all lines in the plane) (i.e., $\parallel$ is reflexive)

S: if $L_j \parallel L_k$, then $L_k \parallel L_j$ (i.e., $\parallel$ is symmetric)

T: if $L_j \parallel L_k$, and $L_k \parallel L_m$, then $L_j \parallel L_m$ (i.e., $\parallel$ is transitive)

(b) $\perp$ is not an equivalence relation

R: L_j is not perpendicular to L_j for all j; (i.e., $\perp$ is nor reflexive)

S: if $L_j \perp L_k$, then $L_k \perp L_j$ (i.e., $\perp$ is symmetric)

T: if $L_j \perp L_k$ and $L_k \perp L_m$, then L_j is not $\perp L_k$ (i.e., $\perp$ is not transitive)

(c) If we include Washington D.C., then ~ (lives in the same state as) is not an equivalence relation DC is not a state and Americans live there.

R: p_j is not ~ p_j for Americans living in DC (~ is not reflexive)

S: if $p_j \sim p_k$, then $p_k \sim p_j$ (i.e., ~ is symmetric)

T: if $p_j \sim p_k$, and $p_k \sim p_m$, then $p_j \sim p_m$ (~ is transitive)

(d) ≈ (lives in the same or an adjacent state) same as (c)

R: p_j is not ≈ p_j for Americans living in DC (≈ is not reflexive)

S: if $p_j \approx p_k$, then $p_k \approx p_j$ (≈ is symmetric)

T: if $p_j \approx p_k$, and $p_k \approx p_m$, then p_j may not be ≈ p_m (≈ is not transitive)

(e.g., let p_j = lives in NJ, p_k = lives in NY, p_m = lives in CT)

(e) ≈ (has a parent in common with) is not an equivalence relation

R: $p_j \approx p_j$ for all j (≈ is reflexive)

S: if $p_j \approx p_k$, then $p_k \approx p_j$ (≈ is symmetric)

T: if $p_j \approx p_k$, and $p_k \approx p_m$, then p_j may not be ≈ p_m (≈ is not transitive)

(e.g., p_j and p_k have a common father, p_k and p_m have a common mother)

(f) ≅ (has the same mother as) is an equivalence relation

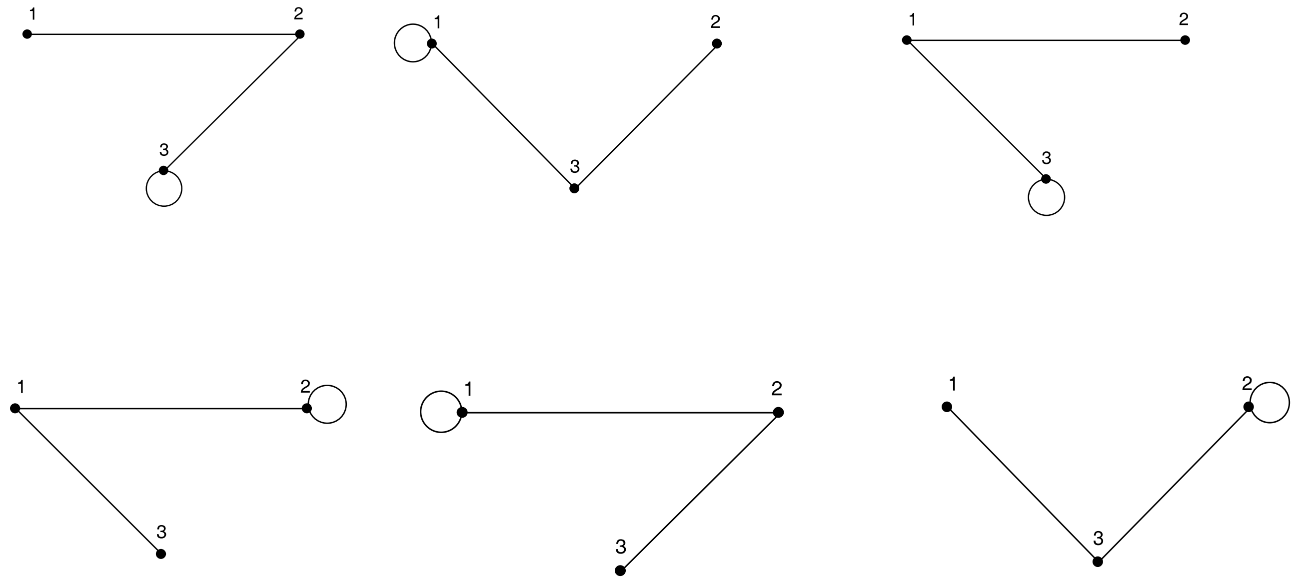
R: $p_j \cong p_j$ for all j (≅ is reflexive)

S: if $p_j \cong p_k$, then $p_k \cong p_j$ (≅ is symmetric)

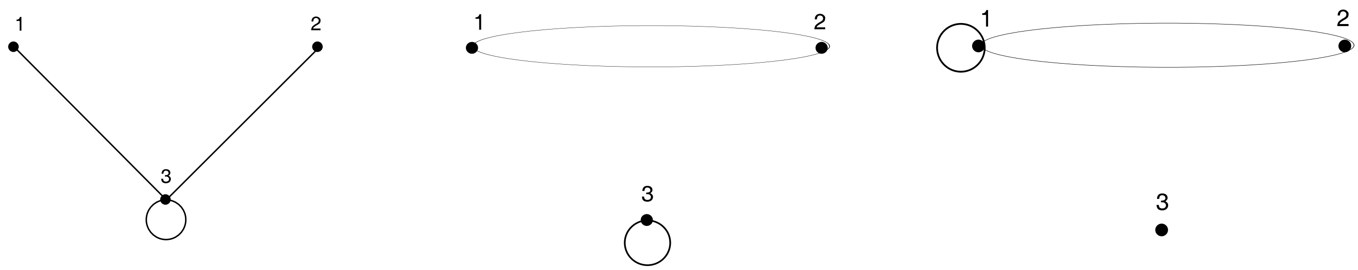
T: if $p_j \cong p_k$, and $p_k \cong p_m$, then $p_j \cong p_m$ (≅ is transitive)

5. G ≃ H

(a) these are isomorphic



(b) and (c)



(d) ≃ is an equivalence relation

R: let $f = 1_{\{1,2,\ldots,n\}}$; hence, f labels G to G (≃ is reflexive)

S: if f labels G to H, then f^{-1} labels H to G (≃ is symmetric)

T: if f labels G to H and g labels H to K, then $g \circ f$ labels G to K (≃ is transitive)

7. For $m,n \in \mathbb{Z}$, $m \sim n$ if $m^2 = n^2$

(a) $\sim$ is an equivalence relation

R: $m \sim m$ since $m^2 = m^2$ $\forall m \in \mathbb{Z}$ ($\sim$ is reflexive)

S: if $m \sim n$, then $n \sim m$ ($\sim$ is symmetric)

(since if $m^2 = n^2$, then $n^2 = m^2$)

T: if $m \sim n$ and $n \sim p$, then $m \sim p$ ($\sim$ is transitive)

(since if $m^2 = n^2$ and $n^2 = p^2$, then $m^2 = p^2$)

(b) There are an infinite number of equivalence classes

$\{0\}$, $\{n,-n\}$ for $n \in \mathbb{P}$

13. For $\mathbb{Z} \times \mathbb{P}$, $(m,n) \sim (p,q)$ if $mq = np$

Note: since $0 \notin \mathbb{P}$, $mq = np$ implies $m/n = p/q$

(a)

R: $(m,n) \sim (m,n)$, since $mn = nm$ ($\sim$ is reflexive)

S: if $(m,n) \sim (p,q)$, then $(p,q) \sim (m,n)$ ($\sim$ is symmetric)

(since if $mq = np$, then $pn = qm$)

T: if $(m,n) \sim (p,q)$ and $(p,q) \sim (s,t)$, then $(m,n) \sim (s,t)$ ($\sim$ is transitive)

(since if $mq = np$ and $pt = qs$, then $m/n = p/q$ and $p/q = s/t$ which implies that $m/n = s/t$. Hence, $mt = ns$ and $(m,n) \sim (s,t)$)

(b) $f: \mathbb{Z} \times \mathbb{P} \to \mathbb{Q}$ where $f(m,n) = m/n$

As above, since $0 \notin \mathbb{P}$, $mq = np$ implies $m/n = p/q$.

Hence, $(m,n) \sim (p,q)$ iff $f(m,n) = f(p,q)$

15. For $m,n \in \mathbb{Z}$, $m \sim n$ if $m^2 = n^2$

(a) $[m] \leq [n]$ iff $m \leq n$ (not well defined)

if the definition made sense $[3] = [-3] \leq [2]$ would imply $3 \leq 2$

(b) $f: [\mathbb{Z}] \to \mathbb{Z}$ with $f([m]) = m^2 + m + 1$ (not well defined)

Note: $[2] = [-2]$; however, $2^2 + 2 + 1 \neq (-2)^2 - 2 + 1$

(c) $g: [\mathbb{Z}] \to \mathbb{Z}$ with $g([m]) = m^4 + m^2 + 1$ (well defined)

Note: $2^4 + 2^2 + 1 = (-2)^4 + (-2)^2 + 1$

(d) Let $[m] \oplus [n] = [m + n]$ (not well defined)

e.g., $[1] = [-1]$; however, $[-1 + 1] = [0] \neq [2] = [1 + 1]$