

Homework Solutions - Section 4.2

1.

Example 1 shows that $n^5 - n$ is a multiple of 5 for $n \in \mathbb{P}$ as follows:

(B) check the basis for $n = 1$

For $n = 1$, $1^5 - 1 = 0$, which is a multiple of 5 (Thus, we have a basis).

(I) Assume that $n^5 - n$ is a multiple of 5 for $n \leq k$, then for $n = k+1$,
 $(k+1)^5 - (k+1) = (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$ which is a multiple of 5.

Now, for $n \in \mathbb{P}$, $n^5 - n$ must be even.

Hence, $n^5 - n$ must be an even multiple of 5; thus, a multiple of 10.

5. By induction:

(B) check the basis for $n = 1$

For $n = 1$, $\sum_{i=1}^n i^2 = \sum_{i=1}^1 i^2 = 1^2 = 1 = \frac{(n)(n+1)(2n+1)}{6} = \frac{(1)(1+1)(2+1)}{6}$

Thus, we have a basis.

(I) Assume, $\sum_{i=1}^n i^2 = \frac{(n)(n+1)(2n+1)}{6}$ for $n \leq k$

Then, for $n = k+1$, we show that the equation still holds, since by the inductive assumption:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^k i^2 + (k+1)^2 = \frac{(k)(k+1)(2k+1)}{6} + (k+1)^2 = \frac{2k^3+9k^2+13k+6}{6} \text{ and}$$

$$\frac{(n)(n+1)(2n+1)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{2k^3+9k^2+13k+6}{6}$$

Thus, the equation still holds.

7.

(a) $37^{100} - 37^{20}$ is of the form $n^5 - n$ (with $n = 37^{20}$), which from exercise 1 is a multiple of 10.

(b) Similarly, $37^{20} - 37^4$ is of the form $n^5 - n$ (with $n = 37^4$), which is a multiple of 10.

(c) $37^{500} - 37^4 = (37^{500} - 37^{100}) + (37^{100} - 37^{20}) + (37^{20} - 37^4)$, each part of which is of the form $n^5 - n$, which is a multiple of 10.

(d) $37^4 - 1 = 1,874,160$ which is a multiple of 10. (Note: The least significant digit of 37^4 must be a 1; hence, $37^4 - 1$ must end in 0 and is thus a multiple of 10).

(e) $37^{500} - 1 = (37^{500} - 37^4) + (37^4 - 1)$, which from (c) and (d) must be a multiple of 10.

13.

(a) Assume, $\sum_{i=0}^k 2^i = 2^{k+1} - 1$ for $k \leq n$,

then for $k = n+1$, $\sum_{i=0}^{n+1} 2^i = \sum_{i=0}^n 2^i + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1$

so the equation holds for the new value of k .

(b) Assume, $\sum_{i=0}^k 2^i = 2^{k+1}$ for $k \leq n$,

then for $k = n+1$, $\sum_{i=0}^{n+1} 2^i = \sum_{i=0}^n 2^i + 2^{n+1} = 2^{n+1} + 2^{n+1} = 2 \cdot 2^{n+1} = 2^{n+2}$

so the equation holds for the new value of k .

(c) By induction:

Part (a) is the inductive step (I). We now check the basis ($k=0$):

(B) for $k = 0$, $\sum_{i=0}^0 2^i = 2^0 = 1 = 2^1 - 1$

Hence, the basis is true and our proposition is true.

(d)

Part (b) is the inductive step (I). We now check the basis ($k=0$):

(B) for $k = 0$, $\sum_{i=0}^0 2^i = 2^0 = 1 \neq 2^1 = 2$

Hence, the basis is false and our proposed proposition is false.

17.

(a) This is the inductive step:

Assume, $p(n) = n^2 + 5n + 1$ is even for $n \leq k$.

Now, for $n = k+1$, $p(k+1) = (k+1)^2 + 5(k+1) + 1 = (k^2 + 5k + 1) + 2k + 6 = p(k) + (2k + 6)$ which must be even if $p(k)$ is even.

(b) Let's test the basis:

For $n = 1$, $p(1) = 1 + 5 + 1 = 7$ which is odd.

Hence, the basis is false, and $p(n)$ is false for all $n \in \mathbb{P}$.

19. By induction:

(B) check the basis for $n = 1$

For $n = 1$, $5^n - 4n - 1 = 5^1 - 4(1) - 1 = 0$ which is divisible by 16.

Thus we have a basis.

(I) Assume, $5^n - 4n - 1$ is divisible by 16 for $n \leq k$.

Then, for $n = k+1$, $5^{k+1} - 4(k+1) - 1 = 5 \cdot 5^k - 4k - 5 = 5(5^k - 4k - 1) + 16k$ which must also be divisible by 16 since $(5^k - 4k - 1)$ is assumed divisible by 16.

Thus, the assumption holds for $n = k + 1$.

Hence, $5^n - 4n - 1$ is divisible by 16 for $n \in \mathbb{P}$.