Homework Solutions - Section 4.5

1. $s_0 = 3$; $s_n = -2s_{n-1}$ for $n \geq 1$
This is of the form $s_n = as_{n-1}$, which has a solution of the form $s_n = a^n s_0$.
Hence, $s_n = (-2)^n s_0 = 3\cdot(-2)^n$ for $n \in \mathbb{N}$
(see problem 3 below)

3. By induction:
(B) check the basis for n = 0
For $n = 0$, $s_0 = a^0 s_0 = 1\cdot s_0 = s_0$ (Hence, we have a basis)
(I) Assume, $s_n = a^n s_0$ for $n \leq k$
Then, for $n = k+1$, $s_{k+1} = as_k = a\cdot a^k s_0 = a^{k+1} s_0$
Thus, the assumption holds for n = k+1 (and we have a proof).

7.
Using theorem 1, the characteristic equation (with a = 1 and b = 2) is:

$$x^2 - x - 2 = (x - 2)(x + 1) = 0$$

The roots are $r_1 = 2$, $r_2 = -1$
Therefore, $s_n = c_1 r_1^{\ n} + c_2 r_2^{\ n}$
Now,

$$s_0 = 3 = c_1 r_1^{\ 0} + c_2 r_2^{\ 0} = c_1 + c_2$$
$$s_1 = 6 = c_1 r_1^{\ 1} + c_2 r_2^{\ 1} = c_1 r_1 + c_2 r_2 = 2c_1 - c_2$$

Solving, we get; $c_1 = 3$ and $c_2 = 0$
Hence, $s_n = 3\cdot r_1^{\ n} + 0\cdot r_2^{\ n} = 3\cdot 2^n$ for $n \in \mathbb{N}$

11.

(a)

Using theorem 1, the characteristic equation (with a = -1 and b = 6) is:

$$x^2 + x - 6 = (x + 3)(x - 2) = 0$$

The roots are $r_1 = -3$, $r_2 = 2$

Therefore, $s_n = c_1 r_1^{\,n} + c_2 r_2^{\,n}$

Now,

$$s_0 = 2 = c_1 r_1^{\,0} + c_2 r_2^{\,0} = c_1 + c_2$$

$$s_1 = -1 = c_1 r_1^{\,1} + c_2 r_2^{\,1} = c_1 r_1 + c_2 r_2 = -3c_1 + 2c_2$$

Solving, we get; $c_1 = 1$ and $c_2 = 1$

Hence, $s_n = 1 \cdot r_1^{\,n} + 1 \cdot r_2^{\,n} = (-3)^n + 2^n$ for $n \in \mathbb{N}$

(b)

As per exercises 1 and 3,

$$s_n = a^n s_0 = 5^n s_0 = 2 \cdot 5^n \text{ for } n \in \mathbb{N}$$

(c)

Using theorem 1, the characteristic equation (with a = 4 and b = -4) is:

$$x^2 - 4x + 4 = (x - 2)(x - 2) = 0$$

The roots are $r_1 = r_2 = 2$

Therefore, $s_n = c_1 r^n + c_2 \cdot n \cdot r^n$

Now,

$$s_0 = 1 = c_1 r^0 + c_2 \cdot 0 \cdot r^0 = c_1$$

$$s_1 = 8 = c_1 r^1 + c_2 \cdot 1 \cdot r^1 = c_1 r + c_2 r = 2c_1 + 2c_2$$

Solving, we get; $c_1 = 1$ and $c_2 = 3$

Hence, $s_n = 1 \cdot r^n + 3 \cdot n \cdot r^n = 2^n + 3 \cdot n \cdot 2^n = (1+3n)2^n$ for $n \in \mathbb{N}$

(d)

Using theorem 1, the characteristic equation (with a = 5 and b = -6) is:

$$x^2 - 5x + 6 = (x - 2)(x - 3) = 0$$

The roots are $r_1 = 2$, $r_2 = 3$

Therefore, $s_n = c_1 r_1^{\,n} + c_2 r_2^{\,n}$

Now,

$$s_0 = c = c_1 r_1^{\,0} + c_2 r_2^{\,0} = c_1 + c_2$$

$$s_1 = d = c_1 r_1^{\,1} + c_2 r_2^{\,1} = c_1 r_1 + c_2 r_2 = 2c_1 + 3c_2$$

Solving, we get; $c_1 = (3c - d)$ and $c_2 = (d - 2c)$

Hence, $s_n = (3c-d) \cdot r_1^{\,n} + (d-2c) \cdot r_2^{\,n} = (3c-d)2^n + (d-2c)3^n$ for $n \in \mathbb{N}$

11. continued

(e) from page 160, with a = 0 and b = 1

$s_{2n} = b^n s_0 = s_0 = 1$ for $n \in \mathbb{N}$

$s_{2n+1} = b^n s_1 = s_1 = 4$ for $n \in \mathbb{N}$

(f) from page 160, with a = 0 and b = 3

$s_{2n} = b^n s_0 = 3^n s_0 = 3^n$ for $n \in \mathbb{N}$

$s_{2n+1} = b^n s_1 = 3^n s_1 = 2 \cdot 3^n$ for $n \in \mathbb{N}$

(g)

Using theorem 1, the characteristic equation (with a = -2 and b = 3) is:

$$x^2 + 2x - 3 = (x + 3)(x - 1) = 0$$

The roots are $r_1 = -3$, $r_2 = 1$

Therefore, $s_n = c_1 r_1^n + c_2 r_2^n$

Now,

$$s_0 = 1 = c_1 r_1^0 + c_2 r_2^0 = c_1 + c_2$$

$$s_1 = -3 = c_1 r_1^1 + c_2 r_2^1 = c_1 r_1 + c_2 r_2 = -3c_1 + c_2$$

Solving, we get; $c_1 = 1$ and $c_2 = 0$

Hence, $s_n = 1 \cdot r_1^n + 0 \cdot r_2^n = (-3)^n$ for $n \in \mathbb{N}$

(h)

Using theorem 1, the characteristic equation (with a = -2 and b = 3) is:

$$x^2 + 2x - 3 = (x + 3)(x - 1) = 0$$

The roots are $r_1 = -3$, $r_2 = 1$

Therefore, $s_n = c_1 r_1^n + c_2 r_2^n$

Now,

$$s_0 = 1 = c_1 r_1^0 + c_2 r_2^0 = c_1 + c_2$$

$$s_1 = 2 = c_1 r_1^1 + c_2 r_2^1 = c_1 r_1 + c_2 r_2 = -3c_1 + c_2$$

Solving, we get; $c_1 = -1/4$ and $c_2 = 5/4$

Thus, $s_n = -(1/4) \cdot r_1^n + (5/4) \cdot r_2^n = -(1/4)(-3)^n + (5/4) = 1/4(5-(-3)^n)$ $\forall n \in \mathbb{N}$

15.

(a)

Using theorem 2, (with A = 3, B = 0, s_1 = 1)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = 2^m \cdot 1 + (2^m - 1) \cdot 3 = 2^{m+2} - 3$

(b)

Using theorem 2, (with A = 0, B = 0, s_1 = 3)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = 2^m \cdot 3 = 3 \cdot 2^m$

(c)

Using theorem 2, (with A = 0, B = 5, s_1 = 0)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = (5/2) \cdot 2^m \cdot m = 5 \cdot 2^{m-1} \cdot m$

(d)

Using theorem 2, (with A = 3, B = 5, s_1 = 2)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = 2^m \cdot 2 + (2^m - 1) \cdot 3 + (5/2) \cdot 2^m \cdot m$

$= 5 \cdot 2^m - 3 + 5 \cdot 2^{m-1} \cdot m$

(e)

Using theorem 2, (with A = -7, B = 0, s_1 = 1)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = 2^m \cdot 1 + (2^m - 1) \cdot -7 = 7 - 6 \cdot 2^m$

(f)

Using theorem 2, (with A = -7, B = 0, s_1 = 5)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = 2^m \cdot 5 + (2^m - 1) \cdot -7 = 7 - 2^{m+1}$

(g)

Using theorem 2, (with A = 0, B = -1, s_1 = 3)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = 2^m \cdot 3 + (-1/2) \cdot 2^m \cdot m = (6 - m) \cdot 2^{m-1}$

(h)

Using theorem 2, (with A = 5, B = -7, s_1 = 0)

$s_{2^m} = 2^m \cdot s_1 + (2^m - 1) \cdot A + (B/2) \cdot 2^m \cdot m = (2^m - 1) \cdot 5 + (-7/2) \cdot 2^m \cdot m$

$= (10 - 7m) \cdot 2^{m-1} - 5$