Homework Solutions - Section 4.6

1.

Prove: $3 + 11 + ... + (8n - 5) = 4n^2 - n$ for $n \in \mathbb{P}$

(B) check the basis for $n = 1$

for $n = 1$; $8n - 5 = 8(1) - 5 = 3$ and $4n^2 - n = 4(1) - 1 = 3$

(we have a basis)

(I) Assume, $3 + 11 + ... + (8n - 5) = 4n^2 - n$ for $n \leq k$,

Then for $n = k + 1$, by the inductive assumption:

$3 + 11 + ... + (8(k+1) - 5) = 3 + 11 + ... + (8k - 5) + (8(k+1) - 5)$

$= 4k^2 - k + (8(k+1) - 5) = 4k^2 + 7k + 3$, and

$4n^2 - n = 4(k+1)^2 - (k+1) = 4k^2 + 7k + 3$

so that the formula holds for $n = k+1$ which proves our proposition.

7.

(a)

$a_0 = a_1 = 1$; $a_n = (a_{n-1}^2 + a_{n-2})/(a_{n-1} + a_{n-2})$

$a_2 = (a_1^2 + a_0)/(a_1 + a_0) = (1^2 + 1)/(1 + 1) = 1$

$a_3 = (a_2^2 + a_1)/(a_2 + a_1) = (1^2 + 1)/(1 + 1) = 1$

$a_4 = (a_3^2 + a_2)/(a_3 + a_2) = (1^2 + 1)/(1 + 1) = 1$

(b)

$a_n = 1 \ \forall n \in \mathbb{N}$

(c)

Prove using the Second Principle of Induction:

(B) check the basis for $n = 0,1$

$a_0 = a_1 = 1$ (we have a basis.)

Note: we need 2 basis points since a_n is a function of a_{n-2}.

(I) Assume, $a_n = 1$ for $n \leq k$,

Then, for $n = k+1$, $a_{k+1} = (a_k^2 + a_{k-1})/(a_k + a_{k-1}) = (1^2 + 1)/(1 + 1) = 1$

so that the equation holds for $n = k+1$ which proves our proposition.

11.

(a)

$a_0 = a_1 = a_2 = 1$; $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

$a_3 = a_2 + a_1 + a_0 = 1 + 1 + 1 = 3$

$a_4 = a_3 + a_2 + a_1 = 3 + 1 + 1 = 5$

$a_5 = a_4 + a_3 + a_2 = 5 + 3 + 1 = 9$

(b)

Prove using the Second Principle of Induction:

(B) check the basis for $n = 0,1,2$

$a_0 = a_1 = a_2 = 1$ are all odd (we have a basis)

Note: we need 3 basis points since a_n is a function of a_{n-3}.

(I) Assume, a_n is odd for $n \leq k$

Then, for $n = k+1$, $a_{k+1} = a_k + a_{k-1} + a_{k-2}$ which is the sum of three odd integers and must be odd.

Thus, the assumption holds for $n = k+1$ which proves our proposition.

(c)

Prove using the Second Principle of Induction:

(B) check the basis for $n = 1,2,3$

$a_1 = 1 \leq 2^{1-1} = 2^0 = 1$; $a_2 = 1 \leq 2^{2-1} = 2^1 = 2$; $a_3 = 3 \leq 2^{3-1} = 2^2 = 4$

Thus, we have a basis.

(I) Assume, $a_n \leq 2^{n-1}$ for $n \leq k$

Then, for $n = k+1$, $a_{k+1} = a_k + a_{k-1} + a_{k-2} \leq 2^{k-1} + 2^{k-2} + 2^{k-3} = 7 \cdot 2^{k-3} < 2^k$

Thus, the assumption holds for $n = k+1$ which proves our proposition.

13.

(a)

$b_0 = b_1 = b_2 = 1$; $b_n = b_{n-1} + b_{n-3}$

$b_3 = b_2 + b_0 = 1 + 1 = 2$

$b_4 = b_3 + b_1 = 2 + 1 = 3$

$b_5 = b_4 + b_2 = 3 + 1 = 4$

$b_6 = b_5 + b_3 = 4 + 2 = 6$

(b)

(B) check the basis for n = 3,4,5

$b_3 = 2 \geq 2 \cdot b_1 = 2 \cdot 1 = 2$; $b_4 = 3 \geq 2 \cdot b_2 = 2 \cdot 1 = 2$; $b_5 = 4 \geq 2 \cdot b_3 = 2 \cdot 2 = 4$

Thus, we have a basis.

(I) Assume, $b_n \geq 2 \cdot b_{n-2}$ for $n \leq k$

Then, for $n = k+1$, $b_{k+1} = b_k + b_{k-2} \geq 2 \cdot b_{k-2} + 2 \cdot b_{k-4} = 2 \cdot b_{k-1}$

Thus, the assumption holds for n = k+1 which proves our proposition.

(c)

(B) check the basis for n = 2,3,4

$b_2 = 1 \geq (\sqrt{2})^{n-2} = (\sqrt{2})^{2-2} = 1$

$b_3 = 2 \geq (\sqrt{2})^{n-2} = (\sqrt{2})^{3-2} = \sqrt{2}$

$b_4 = 3 \geq (\sqrt{2})^{n-2} = (\sqrt{2})^{4-2} = 2$

Thus, we have a basis.

(I) Assume, $b_n \geq (\sqrt{2})^{n-2}$ for $n \leq k$. Then, for $n = k + 1$,

$$b_{k+1} = b_k + b_{k-2} \geq (\sqrt{2})^{k-2} + (\sqrt{2})^{k-4} = 3 \cdot (\sqrt{2})^{k-4} = \frac{3}{2\sqrt{2}}(\sqrt{2})^{k-1} > (\sqrt{2})^{k-1}$$

Thus, the assumption holds for n = k+1 which proves our proposition.