1. a) Find the exact value of arcsec(−2) (no calculator should be used for this, since a calculator can only give approximate answers).

**Solution.** By definition, \( y = \text{arcsec} x \) if \( x = \sec y \) and either \( 0 \leq y < \pi/2 \) or \( \pi \leq y < 3\pi/2 \). Hence, we are looking for a \( y \) for which \( \sec y = -2 \), i.e., \( \cos y = 1/\sec y = -1/2 \). Now \( \cos \pi/3 = 1/2 \). As \( \cos(\pi - t) = -\cos t \), we have \( -1/2 = \cos(\pi - \pi/3) = \cos 2\pi/3 \). However, \( 2\pi/3 \) is in the second quadrant, and we want a \( y \) in the first or third quadrant. As \( \cos(-t) = \cos t \), we can get a number in the third quadrant: \( \cos(-2\pi/3) = \cos(2\pi/3) = -1/2 \). Finally, as \( \cos t = \cos(t + 2\pi) \), \( y = (-2\pi/3) + 2\pi = 4\pi/3 \) will satisfy all the requirements: \( \cos 4\pi/3 = -1/2 \), i.e. \( \sec 4\pi/3 = -2 \), and \( \pi \leq 4\pi/3 < 3\pi/2 \) (this inequality is easy to verify by multiplying through with 6). Hence \( \text{arcsec}(-2) = 4\pi/3 \).

b) Find \( \tan(\arcsin x) \).

**Solution.** Writing \( y = \arcsin x \), we have \( \sin y = x \), and \( \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \) (with the positive value of the square root, since \( -\pi/2 \leq y \leq \pi/2 \), i.e., \( y \) is in the first or fourth quadrant, and the cosine of those angles is nonnegative). Hence
\[
\tan(\arcsin x) = \tan y = \frac{\sin y}{\cos y} = \frac{x}{\sqrt{1 - x^2}}.
\]

c) Calculate the derivative of \( x \arctan x \).

**Solution.** Using the product rule of differentiation, we have
\[
(x \arctan x)' = (x)' \arctan x + x(\arctan x)'
= 1 \cdot \arctan x + x \cdot \frac{1}{1 + x^2} = \arctan x + \frac{x}{1 + x^2}.
\]

2. Set up the integrals to do the following calculations. *Do not calculate any of the integrals.* For each of the parts, consider the region bounded by the curves \( y = x^2 - 4x + 5 \) and \( y = x + 1 \).

a) To find the area of the above region.

**Solution.** First we solve the system equations \( y = x^2 - 4x + 5 \) and \( y = x + 1 \). Since the left-hand sides are equal, we can equate the right-hand sides:
\( x^2 - 4x + 5 = x + 1 \), i.e., \( x^2 - 5x + 4 = 0 \). We can factor the left-hand side: \( (x - 1)(x - 4) = 0 \). The only way for a product to be zero is for at least one of the factors to be zero; that is \( x = 1 \) or \( x = 4 \). This shows that the two curves intersect at the abscissas \( x = 1 \) and \( x = 4 \). The line segment \( y = x + 1 \) is on top, and the parabola \( y = x^2 - 4x + 5 \) is on the bottom. Hence, the area is
\[
A = \int_1^4 (x + 1) \, dx - \int_1^4 (x^2 - 4x + 5) \, dx = \int_1^4 (-x^2 + 5x - 4) \, dx.
\]

b) To find the volume obtained by rotating the above region about the \( x \) axis by using the method of slices (also called the method of disks, washers, or cross sections).

**Solution.** The formula for the volume \( V \) of the solid obtained by rotating the region between abscissas \( a \) and \( b \) and under the curve \( y = f(x) \) about the \( x \) axis is \( V = \int_a^b \pi(f(x))^2 \, dx \); this formula is obtained by using the method of slices. In the present case,
\[
V = \int_1^4 \pi(x + 1)^2 \, dx - \int_1^4 \pi(x^2 - 4x + 5)^2 \, dx = \int_1^4 \pi(-x^4 + 8x^3 - 25x^2 + 42x - 24) \, dx.
\]

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1 All computer processing for this manuscript was done under Fedora Linux. AMSTeX was used for typesetting. PCTeX was used for the diagrams, together with the programming language Perl used for generating the data points.
c) To find the volume obtained by rotating the above region about the y axis by using the method of cylindrical shells.

**Solution.** The formula for the volume $V$ of the solid obtained by rotating the region between abscissas $a$ and $b$ and under the curve $y = f(x)$ about the $y$ axis is $\int_a^b 2\pi x f(x) \, dx$; this formula is obtained by using the method of cylindrical shells. In the present case,

$$V = \int_1^4 2\pi x(x+1) \, dx - \int_1^4 2\pi x(x^2 - 4x + 5) \, dx = \int_1^4 2\pi (-x^3 + 5x^2 - 4x) \, dx.$$

3. Calculate the integrals

a) $\int_0^{\pi/2} x^2 \cos x \, dx$,

b) $\int_0^{\pi/4} \tan^5 x \sec^2 x \, dx$.

**Solution.** As for a), we need to use integration by parts twice:

$$\int_0^{\pi/2} x^2 \cos x \, dx = \left[ x^2 \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x \, dx = \frac{\pi^2}{4} + \int_0^{\pi/2} 2x \cos x \, dx = \frac{\pi^4}{4} + 2 \int_0^{\pi/2} \cos x \, dx = \frac{\pi^4}{4} - 2.$$

As for b), we need to use the substitution $t = \tan x$. Then $dt = \sec^2 x \, dx$; if $x = 0$ then $t = 0$, and if $x = \pi/4$ then $t = 1$. Hence

$$\int_0^{\pi/4} \tan^5 x \sec^2 x \, dx = \int_0^1 t^5 \, dt = \frac{1}{6} \left[ \frac{t^6}{6} \right]_0^1 = \frac{1}{6}.$$

4. Calculate the integral

$$\int \frac{1}{x^2 \sqrt{x^2 + 1}} \, dx.$$

**Solution.** We will use the substitution $t = \arctan x$. In this case we have $x = \tan t$, $dx = \sec^2 t \, dt$ and $\sqrt{x^2 + 1} = \sqrt{\tan^2 t + 1} = \sec t$; note that the right-hand side is $+ \sec t$ and not $\pm \sec t$, since we take the nonnegative value of the square root, and, further, the choice of $t = \arctan x$ means, by the definition of arctan that $t$ is in the interval $(-\pi/2, \pi/2)$, and $\sec t$ is positive in this interval. Thus,

$$I = \int \frac{1}{x^2 \sqrt{x^2 + 1}} \, dx = \int \frac{1}{\tan^2 t \sec t} \, dt = \int \frac{\sec t}{\tan^2 t \sec^2 t} \, dt = \int \frac{\sec t}{\sin^2 t} \, dt = \int \frac{\cos t \, dt}{\sin^2 t}.$$

To calculate this integral, we need to substitute $u = \sin t$, when $du = \cos t \, dt$. That is,

$$I = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\sin t} + C = -\frac{1}{\tan t \cos t} + C = -\frac{\sec t}{\tan t} + C = -\frac{\sqrt{x^2 + 1}}{x} + C.$$

5. Calculate the integral

$$\int \frac{x - 5}{x^4 + 4x^3 + 5x^2} \, dx.$$

**Solution.** We need to use partial fraction decomposition. Noting that the denominator of the integrand can be factored as $x^2(x^2 + 4x + 5)$, and $x^2 + 4x + 5 = (x + 2)^2 + 1$ has no real zeros, we have

$$\frac{x - 5}{x^4 + 4x^3 + 5x^2} = \frac{x - 5}{x^2(x^2 + 4x + 5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 2x + 5}.$$
that is, multiplying both sides by \( x^2(x^2 + 2x + 5) \), we obtain

\[
x - 5 = Ax(x^2 + 4x + 5) + B(x^2 + 4x + 5) + (Cx + D)x^2.
\]

Note that while the former equation does not make sense for \( x = 0 \), the latter equation holds even in this case, since if two polynomials agree at infinitely many points, they must agree at every point (since, by rearranging the equation, we will get a polynomial equation with infinitely many solutions, and a polynomial that is not identically zero can only have finitely many zeros).

Substituting \( x = 0 \) in the latter equation gives \(-5 = 5B\), that is, \( B = -1 \). Next, we will differentiate this equation and then substitute \( x = 0 \). We obtain

\[
1 = Ax(x^2 + 4x + 5)' + A(x^2 + 4x + 5) + B(2x + 4) + (Cx + D)'x^2 + (Cx + D)2x;
\]

it unnecessary to calculate the indicated derivatives, since when we substitute \( x = 0 \) the coefficients of \( x \) will be multiplied by zero.\(^3\) After substituting \( x = 0 \) we obtain \( 1 = 5A + 4B \). Noting that \( B = -1 \), This gives \( A = 1 \). Equating the coefficients of \( x^3 \) on both sides of the (original, undiffereniated) equation, we obtain \( 0 = A + C \); given that \( A = 1 \), this gives \( C = -1 \). Finally, equating the coefficients of \( x^2 \), we obtain \( 0 = 4A + B + D \). Noting that \( A = 1 \) and \( B = -1 \), this gives \( D = -3 \). That is,

\[
\frac{x - 5}{x^4 + 4x^3 + 5x^2} = \frac{1}{x} - \frac{1}{x^2} - \frac{x + 3}{x^2 + 2x + 5},
\]

As for integrating this, the first two terms are easy to integrate. For the last term, we have

\[
\int \frac{x + 3}{x^2 + 4x + 5} \, dx = \int \frac{x + 2}{x^2 + 4x + 5} \, dx + \int \frac{1}{x^2 + 4x + 5} \, dx.
\]

The first integral is easy to handle with the substitution \( t = x^2 + 4x + 5 \). Then \( dt = 2(x + 2) \, dx \), and so

\[
\int \frac{x + 2}{x^2 + 2x + 5} \, dx = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t| + C = \frac{1}{2} \ln |x^2 + 4x + 5| + C = \frac{1}{2} \ln(x^2 + 4x + 5) + C.
\]

The absolute value after the logarithm is not needed, since \( x^2 + 4x + 5 \) is always positive. As for the latter integral, using the substitution \( t = x + 2 \), when \( dt = dx \), we obtain

\[
\int \frac{1}{x^2 + 4x + 5} \, dx = \int \frac{1}{(x^2 + 2)^2 + 1} \, dx = \int \frac{1}{t^2 + 1} \, dt = \arctan t + C = \arctan(x + 2) + C,
\]

where \( C \) is an arbitrary constant, not necessarily the same as the \( C \) in the previous integral. Putting these together,

\[
\int \frac{x + 2}{x^2 + 2x + 5} \, dx = \frac{1}{2} \ln(x^2 + 4x + 5) + \arctan(x + 2) + C.
\]

Hence,

\[
\int \frac{x - 5}{x^4 + 4x^3 + 5x^2} \, dx = \int \frac{1}{x} \, dx - \int \frac{1}{x^2} \, dx - \int \frac{x + 3}{x^2 + 4x + 5} \, dx
\]

\[
= \ln |x| + \frac{1}{x} - \frac{1}{2} \ln(x^2 + 4x + 5) - \arctan(x + 2) + C,
\]

where \( C \) is an arbitrary constant.

\(^3\)In fact, since our intention was to substitute \( x = 0 \) in the derivative, instead of the last equation we could more simply have written that

\[
1 = A(x^2 + 4x + 5) + B(2x + 4) + x \cdot \text{some polynomial}.
\]