1.a) Decide whether each of the following is true or false. Do not explain.

(i) \( \{1, 3, 5\} = \{3, 1, 5, 3\} \),
(ii) \( \emptyset \in \{\emptyset\} \),
(iii) \( \emptyset \subseteq \{3, 4\} \),
(iv) \( \{2\} \in \{\{2\}\} \),
(v) \( \{2\} \subseteq \{\{2\}\} \),
(vi) for any sets \( A \) and \( B \), \( A \cup B \subseteq A \cap B \),
(vii) for any sets \( A \) and \( B \), \( A \cap B \subseteq A \cup B \),
(viii) for any set \( A \), \( A \cap \emptyset = A \),
(ix) for any set \( A \), \( A \cup \emptyset = A \),
(x) for any sets \( A, B, \) and \( C \), \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

**Solution.** (i) is true. Sets do not have multiple elements. The elements of the set are those listed; if an element is listed more than once, that does not make any difference. Two sets are equal if they have the same element.

(ii) is false. The only element of the set \( \{\emptyset\} \) is \( \emptyset \), and \( \emptyset \neq \emptyset \).

(iii) is true. The empty set is a subset of every set. \( A \subseteq B \iff (\forall x)[x \in A \implies x \in B] \), and the right-hand side here is vacuously true if \( A \) is the empty set, since there is no \( x \) for which \( x \in \emptyset \).

(iv) is true. The only element of \( \{\{2\}\} \) is \( \{2\} \).

(v) is false. \( 2 \) is an element of \( \{2\} \), but it is not an element of \( \{\{2\}\} \); the only element of the latter is \( \{2\} \), and \( 2 \neq \{2\} \).

(vi) is false. If \( x \in A \) but \( x \notin B \), then \( x \in A \cup B \) but \( x \notin A \cap B \).

(vii) is true. The elements of \( A \cap B \) are the those that are elements both of \( A \) and \( B \), and the elements of \( A \cup B \) are those that are elements of either \( A \) or \( B \) (or both), so every element of \( A \cap B \) is also an element of \( A \cup B \).

(viii) is false. \( A \cap \emptyset = \emptyset \), so \( A \cap \emptyset \neq A \) unless \( A = \emptyset \).

(ix) is true. The elements of \( A \cup \emptyset \) are those that are elements of \( A \) or of \( \emptyset \). The set \( \emptyset \) has no elements, so adding its elements to \( A \) will make no difference.

(x) is true. For all \( x \) we have
\[
x \in A \cup (B \cap C) \iff [x \in A \lor (x \in B \land x \in C)],
\]
and
\[
x \in (A \cup B) \cap (A \cup C) \iff [(x \in A \lor x \in B) \land (x \in A \lor x \in C)],
\]
and the right-hand sides of these are equivalent, since
\[
P \lor (Q \land R) \iff (P \lor Q) \land (P \lor R)
\]
is a tautology.

b) For any real number \( x \), let \( A_x \) be the interval \((-\infty, x)\). Describe the sets (i) \( \bigcup_{x \in (-\infty, +\infty)} A_x \), and (ii) \( \bigcap_{x \in (-\infty, +\infty)} A_x \).

**Solution.** A brief reflection shows that (i) \( \bigcup_{x \in (-\infty, +\infty)} A_x = (-\infty, +\infty) \), since every real number \( r \) belongs to (i.e., is an element of) at least one of the sets whose union is taken on the left; in fact, \( r \in A_{r+1} \), for example.

It is also easy to see that (ii) \( \bigcap_{x \in (-\infty, +\infty)} A_x = \emptyset \), since for every real number \( r \) there is at least one set among those whose intersection is taken to which \( r \) does not belong. For example, \( r \notin A_{r+1} \).

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1. All computer processing for this manuscript was done under Fedora Linux. A\textsc{ms-te\textregistered}X was used for typesetting.
2. a) Write the truth table for the formula \((P \land \sim Q) \implies R\)

Solution.

<table>
<thead>
<tr>
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b) Let \(x\) and \(y\) run over integers. Decide whether each of the following is true or false:

i) \((\forall x)(\exists y)[x \neq y]\).

determine condition for which the above statement is true or false. If the assertion is false, also give an example for

ii) \((\exists y)(\forall x)[x \neq y]\).

Explain.

Solution. i) is true. Given an arbitrary integer \(x\), we can pick \(y = x + 1\) to ensure that \(x \neq y\). ii) is not true. To see this, pick an arbitrary number \(y\). Then \((\forall x)[x \neq y]\) is certainly not true; for example, \(x \neq y\) is not true with \(x = y\).

c) Move the negation all the way inside in the formula

\[\sim (\forall x)(\exists y)(\forall z)[P(x) \land (Q(y) \implies \sim R(z))].\]

Solution. We have

\[\sim (\forall x)(\exists y)(\forall z)[P(x) \land (Q(y) \implies \sim R(z))] \equiv (\exists x)\sim (\exists y)(\forall z)\sim [P(x) \land (Q(y) \implies \sim R(z))]
\equiv (\exists x)\sim [P(x) \lor \sim (Q(y) \implies \sim R(z))]
\equiv (\exists x)\sim [P(x) \lor (\sim Q(y) \land \sim R(z))]
\equiv (\exists x)(\sim Q(y))(\exists z)[P(x) \lor (\sim R(z))]
\equiv (\exists x)(\sim Q(y))(\exists z)[P(x) \lor \sim (\sim R(z))]
\equiv (\exists x)(\sim Q(y))(\exists z)[P(x) \lor (Q(y) \land R(z))].\]

The first two lines use the equivalences \(\sim (\exists t) \equiv (\forall t) \sim\) and \(\sim (\forall t) \equiv (\exists t) \sim\), the third line use the De Morgan identity \(\sim (P \land Q) \equiv \sim P \lor \sim Q\) and the equivalence \(P \implies Q \equiv \sim P \lor Q\), the fourth line uses the De Morgan identity \(\sim (P \lor Q) \equiv \sim P \land \sim Q\) and then it uses the law of double negation \(\sim \sim P \equiv P\), and the fifth line again uses the \(\sim (P \land Q) \equiv \sim P \lor \sim Q\) (in the reverse direction).

3. Decide whether each of the following logical equivalences is true or false. In each case, explain why the equivalence in question is true, or why it is false. If the assertion is false, also give an example for \(P(x)\) and \(Q(x)\) that makes the assertion false.

Note: There is a subtle difference between logical equivalence \(\equiv\) and the biconditional \(\iff\). Logical equivalence \(\equiv\) means that the formulas on the two sides have exactly the same meaning, while the biconditional \(\iff\) means that the formulas in a specific realization – when \(x\) ranges over integers, say, and \(P(x)\) and \(Q(x)\) are chosen in a specific way – the two sides are true at the same time.

a) \((\exists x)[P(x) \lor Q(x)]\) \(\equiv\) \(\left(\exists x\right)\left[P(x) \lor (\exists x)Q(x)\right]\),

b) \((\exists x)[P(x) \implies Q(x)]\) \(\equiv\) \(\left(\forall x\right)\left[P(x) \implies (\exists x)Q(x)\right]\),

c) \((\exists x)[P(x) \implies Q(x)]\) \(\equiv\) \(\left(\exists x\right)\left[P(x) \implies (\exists x)Q(x)\right]\),
Solution to Part a) This logical equivalence is true. If \( P(x) \) or \( Q(x) \) is true for at least one \( x \), then both sides are true. If, on the other hand \( P(x) \) and \( Q(x) \) is never true, then both sides are false.

Solution to Part b) This logical equivalence is true. To show this, we distinguish two cases. 1) \((\forall x)P(x)\) is false. 2) \((\forall x)P(x)\) is true. As for Case 1, the left-hand side is true in this case, since picking an arbitrary \( x \) for which \( P(x) \) is false, the conditional \( P(x) \implies Q(x) \) is true. The right-hand side is also true, since the antecedent is false.

As for Case 2), if the right-hand side is true, then there must be an \( x_0 \) for which \( Q(x_0) \) is true. So the conditional \( P(x_0) \implies Q(x_0) \) is true, i.e., the left-hand side is also true. If the right-hand side is false, then for all \( x \) the statement \( Q(x) \) is false, so the conditional \( P(x) \implies Q(x) \) is also false (since \( P(x) \) is true).

This happens for every \( x \), so the left-hand side is false.

A more formal demonstration is the above statement is as follows:

\[
(\exists x)(P(x) \implies Q(x)) \equiv (\exists x)(\sim P(x) \lor Q(x))
\]

\[
\equiv (\exists x)(\sim P(x)) \lor (\exists x)Q(x)
\]

\[
\equiv (\forall x)P(x) \lor (\exists x)Q(x)
\]

\[
\equiv (\forall x)P(x) \implies (\exists x)Q(x);
\]

here the second equivalence is essentially the same as the one expressed by Part a) of this problem; to wit, the expression on its left-hand side is true exactly if there is an \( x \) for which either \( \sim P(x) \) or \( Q(x) \) is true, and the expression on its right-hand side is true exactly at the same time.

Solution to Part c) This equivalence is false. For example, assuming that \( x \) ranges over integers, the left-hand side

\[
(\exists x)(x = 2 \implies x \neq x)
\]

is true, since if we choose \( x = 3 \) then the antecedent \( x = 2 \) of the conditional is false, so the conditional is true. On the other hand

\[
(\exists x)(x = 2) \implies (\exists x)(x \neq x)
\]

is false, since the antecedent \((\exists x)(x = 2)\) is true, while the consequent \((\exists x)(x \neq x)\) is false.

4. a) Define what it means for an integer to be (i) even, and (ii) odd. (Give two separate definitions, one for (i) and one for (ii).)

**Definition (i).** An integer \( n \) is called even if \( n = 2k \) for some integer \( k \).

Formally,

\[
\text{even}(n) \iff (\exists k \in \mathbb{Z})[n = 2k].
\]

In answer to the problem, you need to give the verbal definition, and not the formal definition. Note that the verbal definition says "if", while the formal definition uses \( \iff \). In mathematics, it is very important to say "if" when one means "if", and say "if and only if" when one means the latter. The only exception is the above use of "if" in definitions, where, by tradition, one says "if" but one means "if and only if".

**Definition (i).** An integer \( n \) is called odd if \( n = 2k + 1 \) for some integer \( k \).

Formally,

\[
\text{odd}(n) \iff (\exists k \in \mathbb{Z})[n = 2k + 1].
\]

b) Prove the following: If \( n \) is an odd integer, then \( n^2 - 4n + 5 \) is even. **Proof.** Let \( n \) be an arbitrary odd integer; then there is an integer \( k \) for which \( n = 2k + 1 \). Then

\[
n^2 - 4n + 3 = (2k + 1)^2 - 4(2k + 1) + 5 = (4k^2 + 4k + 1) - (8k + 4) + 5
\]

\[
= 4k^2 - 4k + 2 = 2(2k^2 - 2k + 1).
\]

As \( 2k^2 - 2k + 1 \) is an integer, this shows that \( n^2 - 4n + 5 \) is even, as we wanted to prove. □
c) Prove the following: For every integer $n$, if $5n - 3$ is even, then $n$ is odd.

Proof. Assume $n$ is not odd, that is, assume that $n$ is even. Then we need to show that $5n - 3$ is not even, i.e., it is odd.\(^2\) So, $n$ being even, we have $n = 2k$ for some integer $k$. Now
\[
5n - 3 = 5(2k) - 3 = 10k - 3 = 2(5k - 2) + 1.
\]

As $5k - 1$ is an integer, this shows that $5n - 3$ is odd, as we wanted to show. \(\square\)

5. Prove the following: If $n$ is an integer such that $5n^2 - 2n$ is even, then $3n - 1$ is odd.

Solution. Assuming $5n^2 - 2n$ is even, we will first prove that $n$ is even, and the we will prove that if $n$ is even, then $3n - 1$ is odd. To prove the first statement, we will in fact prove its contrapositive. However, it is better to formulate the proof as a proof by contradiction, since then we can maintain the assumption that $5n^2 - 2n$ is even all the way throughout the proof. In a proof by contrapositive, this assumption would not be made.

Proof. Assume that $5n^2 - 2n$ is even.

Claim 1. $n$ is even.

Proof of Claim 1. Assume, on the contrary, that $n$ is odd; that is, $n = 2k + 1$ for some integer $k$. Then we have
\[
5n^2 - 2n = 5(2k + 1)^2 - 2(2k + 1) = 5(4k^2 + 4k + 1) - 4k - 2
= 20k^2 + 16k + 3 = 2(10k^2 + 8k + 1) + 1.
\]

Since $10k^2 + 8k + 1$ is an integer, this shows that $5n^2 - 2n$ is odd; this is, however, a contradiction, since we assumed that $5n^2 - 2n$ is even. The proof of Claim 1 is complete.

We can now complete the proof as follows. According to Claim 1, $n$ is even; that is, $n = 2k$ for some integer $k$.\(^3\) Then
\[
3n - 1 = 3(2k) - 1 = 6k - 1 = 2(3k - 1) + 1.
\]

Since $3k - 1$ is an integer, this shows that $3n - 1$ is odd. The proof is complete. \(\square\)

\(^2\)This is proving by contrapositive, that is when wanting to prove the conditional $P \implies Q$, we instead prove the equivalent conditional $\sim Q \implies \sim P$. But the method of proof also fits into the larger pattern of proof by contradiction. That is, assuming that the conclusion (here: $n$ is even) is not true, we arrive at a contradiction, that is, at a statement that we know cannot be true (in this case, we will show that the assumption that $3n - 5$ is odd is not true; however, the assumptions of the statement must be true). The only way this can happen is that the assumption (namely that the conclusion is not true), is not correct, since from correct assumptions with correct argument we cannot get an incorrect result.

\(^3\)Observe that we are reusing the letter $k$. $k$ was used only in the proof of Claim 1, and outside that proof, $k$ has no meaning. The phrase “for some integer $k$” directly indicates that $k$ has a new meaning.