1. Let $A$ and $B$ be sets, let $i_A$ and $i_B$ be the identity functions on $A$ and $B$, respectively, and let $f : A \to B$ and $g : B \to A$ be functions. Assume that $f \circ g = i_B$ and $g \circ f = i_A$. Prove that $g$ is the inverse of $f$.

**Proof.** To show that $g$ is the inverse of $f$, we need to show that for arbitrary $a \in A$ and $b \in B$ we have $f(a) = b$ if and only if $g(b) = a$. To show the “if” part, assume that $g(b) = a$; then we need to show that $f(a) = b$. Indeed, we have $g(b) = g(f(a)) = (g \circ f)(a) = id_A(a) = a$. To show the “only if” part, assume that $g(b) = a$; we then need to show that $f(a) = b$. Similarly as before, we have $f(a) = f(g(b)) = (f \circ g)(b) = id_A(b) = b$. \(\square\)

b) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{pmatrix}$ be permutations. Find (i) $\alpha^{-1}$ and (ii) $\alpha \circ \beta$.

**Solution.** (i) To find $\alpha^{-1}$, note that $\alpha(2) = 1$, so $\alpha^{-1}(1) = 2$; $\alpha(5) = 2$, so $\alpha^{-1}(2) = 5$; $\alpha(1) = 3$, so $\alpha^{-1}(3) = 1$; $\alpha(3) = 4$, so $\alpha^{-1}(4) = 3$; $\alpha(4) = 5$, so $\alpha^{-1}(5) = 4$. Thus, $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$.

(ii) To find $\alpha \circ \beta$, note that $(\alpha \circ \beta)(1) = \alpha(\beta(1)) = \alpha(2) = 1$, $(\alpha \circ \beta)(2) = \alpha(\beta(2)) = \alpha(4) = 5$, $(\alpha \circ \beta)(3) = \alpha(\beta(3)) = \alpha(5) = 2$, $(\alpha \circ \beta)(4) = \alpha(\beta(4)) = \alpha(3) = 4$, $(\alpha \circ \beta)(5) = \alpha(\beta(5)) = \alpha(1) = 3$. Thus, $\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix}$.

2. Prove that for $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for every positive integer $n$.

**Proof.** For $n = 1$ both sides are equal to $1/2$, so the assertion is true in this case. Let $n > 1$ and assume that the assertion is true in this case with $n − 1$ replacing $n$, that is

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \frac{n-1}{n}.$$ 

Then

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1} \frac{1}{k(k+1)} + \frac{1}{n(n+1)} = \frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1)+1}{n(n+1)} = \frac{n^2-1+1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1},$$

as we wanted to show. \(\square\)

**Note.** Another way to establish the result is to observe that the sum in question is a telescoping sum:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$
While this proof is much simpler than the one given above, it is somewhat more difficult to find. While the first proof is completely routine, the second one depends on an observation that is easily missed.

b) Prove that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for all $n \geq 0$.

Proof. For $n = 0$ both sides equal 1, so the result is true in this case. Let $n > 1$, and assume that the result is true with $n-1$ replacing $n$, that is

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$ We then have

$$\sum_{k=0}^{n} 2^k = \sum_{k=0}^{n-1} 2^k + 2^n = (2^n - 1) + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1,$$

as we wanted to show. □

3. a) Prove that for every positive integer $n$ and every real $x \geq -1$ we have $(1 + x)^n \geq 1 + nx$. Clearly indicate how the assumption $x \geq -1$ is used in the proof.

Proof. The assertion, called Bernoulli’s inequality, is true for $n = 1$, because then it states that $1 + x \leq 1 + x$. Let $n \geq 1$, and assume the assertion is true for $n$, that is

$$(1 + x)^n \geq 1 + nx$$

for all $x > -1$. Then, for any $x > -1$ we have

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x,$$

as we wanted to show. Note that the first inequality here results by multiplying both sides of the induction hypotheses by $1 + x$. This is allowed, since $1 + x$ is nonnegative (multiplying an inequality by a negative number reverses the inequality). □

b) Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers such that $a_0 = 1$, $a_1 = 2$, and $a_n = 6a_{n-2} - a_{n-1}$ for every integer $n \geq 2$. Prove that $a_n = 2^n$ for every nonnegative integer $n$.

Proof. The assertion is true for $n = 0$ and $n = 1$. Let $n \geq 2$, and assume that the assertion is true for every nonnegative $k < n$ replacing $n$. In particular, $a_{n-1} = 2^{n-1}$ and $a_{n-2} = 2^{n-2}$. Then

$$a_n = 6a_{n-2} - a_{n-1} = 6 \cdot 2^{n-2} - 2^{n-1} = 4 \cdot 2^{n-2} + 2 \cdot 2^{n-2} - 2^{n-1} = 2^n + 2^{n-1} - 2^{n-1} = 2^n,$$

showing that the assertion is also true for $n$. Therefore, the assertion is true for every $n \geq 0$. □

4. a) Define what it means for the set $A$ and $B$ to be equinumerous.

Definition. The sets $A$ and $B$ are called equinumerous if there is a one-to-one function $f : A \to B$ onto $B$.

b) Prove that the set $\mathbb{R}$ of real numbers and the interval $(-1, 1)$ are equinumerous.

Solution. The function $f : (-1, 1) \to \mathbb{R}$ such that

$$f(x) = \frac{x}{x^2 - 1} \quad \text{for} \quad x \in (-1, 1)$$

\[ ^3 \text{Note that one can only do induction on } n, \text{ and not on } x, \text{ since } x \text{ is a real number, not necessarily an integer.} \]
is one-to-one and onto \( \mathbb{R} \). Indeed, let \( y \in \mathbb{R} \) be arbitrary. We need to show that there is exactly one \( x \in (-1, 1) \) such that \( f(x) = y \). If \( y = 0 \) then we have \( f(x) = y \) only for \( x = 0 \). Assume now that \( y \neq 0 \). Then the equation \( f(x) = y \) can be equivalently written as \( y(x^2 - 1) = x \).

Observe that this latter equation makes sense for \( x = \pm 1 \) while the equation \( f(x) = y \) does not. The important point, however, is that \( x = \pm 1 \) does not satisfy the latter equation, since for this choice of \( x \) the left-hand side is 0 while the right-hand side is \( \pm 1 \). That is, the exceptional case of \( x = \pm 1 \) does not affect the equivalence of the two equations.

Keeping in mind that we assumed that \( y \neq 0 \), the latter equation can also be written as

\[
x^2 - \frac{1}{y}x - 1 = 0.
\]

This is a quadratic equation for \( x \). We can solve this equation for \( x \) as

\[
x = \frac{\frac{1}{y} \pm \sqrt{\left(\frac{1}{y}\right)^2 + 4}}{2}.
\]

Given that the discriminant (the expression under the square root) of this equation is positive, this equation has two distinct real solutions; call them \( x_1 \) and \( x_2 \). The product of these two solutions is the constant term of the equation; that is, \( x_1 x_2 = -1 \). Therefore \( |x_1||x_2| = 1 \). Given that \( |x_1|, |x_2| \neq 1 \), as we remarked above, one of \( x_1 \) and \( x_2 \) must be inside the interval \((-1, 1)\) and the other one must be outside. That is, there is exactly one \( x \in (-1, 1) \) for which \( f(x) = y \), as we wanted to show.

5. Show that the intervals \( A = (0, 1) \) and \( B = (0, 1] \) are equinumerous by explicitly describing a one-to-one mapping \( h : A \to B \) onto \( B \). (Note that the mappings \( f : A \to B \) defined by \( f(x) = x \) for \( x \in A \) and \( g : B \to A \) defined by \( g(x) = x/2 \) for \( x \in B \) are one-to-one, so such a mapping \( h \) can be constructed by using the proof of the Cantor–Schröder–Bernstein theorem.)

**Solution.** For \( x \in (0, 1) \) put

\[
h(x) = \begin{cases} 
2x & \text{if } x = 2^{-k} \text{ for some integer } k \geq 1, \\
x & \text{otherwise}.
\end{cases}
\]

It is easy to see that \( h \) is one-to-one and onto \((0, 1]\); in fact, given the above functions \( f \) and \( g \), this is the function \( h \) that is described in the first proof of the Cantor–Schröder–Bernstein theorem in the notes for the present course, with the functions \( f \) and \( g \) described in the problem. To see this, note that the definition of \( h \) could have been written as

\[
h(x) = \begin{cases} 
g^{-1}(x) & \text{if } x = 2^{-k} \text{ for some integer } k \geq 1, \\
f(x) & \text{otherwise}.
\end{cases}
\]

If for \( x \in A = (0, 1) \) one considers the sequence \( x, g^{-1}(x), f^{-1}(g^{-1}(x)), g^{-1}(f^{-1}(g^{-1}(x))), \ldots \) discussed in the proof of the Cantor–Schröder–Bernstein theorem just mentioned, this sequence ends in the set \( B = (0, 1] \) (with the value 1) in Case 1 in the definition of \( h \), and in the set \( A \) in Case 2. Indeed, in Case 1 this sequence is \( 2^{-k} \in A, 2^{-k+1} \in B, 2^{-k+1} \in A, 2^{-k+2} \in B, 2^{-k+2} \in A, \ldots, 2^{-1} \in B, 2^{-1} \in A, 1 \in B \), and in Case 2 it is \( x \in A, 2x \in B, 2x \in A, 2^2x \in B, 2^2x \in A, \ldots, 2^{l-1}x \in B, 2^{l-1}x \in A, 2^{l}x \in B, 2^{l}x \in A \) with \( l \geq 0 \) such that \( 1/2 < 2^{l}x < 1 \).

The function \( \bar{h} \) defined for \( x \in (-1, 1) \) by

\[
\bar{h}(x) = \begin{cases} 
h(x) & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-h(-x) & \text{if } x < 0
\end{cases}
\]

is a one-to-one mapping from \((-1, 1)\) onto \([-1, 1]\). It turns out that there is really no simpler way to describe a one-to-one mapping from \((-1, 1)\) onto \([-1, 1]\).

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4See [http://www.sci.brooklyn.cuny.edu/~mate/intro_proofs/cardinalities.pdf](http://www.sci.brooklyn.cuny.edu/~mate/intro_proofs/cardinalities.pdf) for the temporary link used for the present course, or [http://www.sci.brooklyn.cuny.edu/~mate/misc/cardinalities.pdf](http://www.sci.brooklyn.cuny.edu/~mate/misc/cardinalities.pdf) for a permanent link.