THE UNIQUENESS OF THE ROW ECHelon FORM

Let $M$ be a matrix. A matrix $M'$ is called a row echelon form of $M$ if the following conditions are satisfied.

(i) $M'$ is obtained from $M$ by a finite number of the following three operations, called elementary row operations: 1) interchange of two rows, 2) multiplying a row by a nonzero scalar, and 3) adding a scalar multiple of a row to another row.

(ii) Each row of $M'$ starts with either with a 1, or with a number of zeros followed by a 1, or the row consists entirely of zeros. The first nonzero entry in a row of $M'$ is called the leading entry of that row; according to what we said, this leading entry must be 1.

(iii) If $l > k > 0$, and row $l$ in $M'$ has as a nonzero entry, then row $k$ must also have a nonzero entry, and the leading entry of row $k$ must occur earlier than the leading entry of row $l$. In particular, this means that all purely zero rows must occur at the bottom of matrix $M'$.

(iv) If a column of $M'$ contains a leading entry (of a row), then all other entries in this column must be 0.

Theorem. The row echelon form of a matrix is unique.

Proof. In the proof, we will need the following notation. If a matrix $M$ has at least $n$ columns, write $M \upharpoonright n$ for the submatrix resulting from $M$ by deleting all columns after the $n$th column (in particular, if $M$ has exactly $n$ columns then $M \upharpoonright n = M$). $M \upharpoonright n$ is called the restriction of $M$ to $n$ columns.

The assertion says that a matrix $M$ cannot have two different row echelon forms. Assume, on the contrary that both $M_1$ and $M_2$ are row echelon forms of $M$, and $M_1 \neq M_2$. First notice that, in a row echelon form of $M$, a column consists of all zeros if and only if the corresponding column in $M$ consists only of zeros; this is because the elementary row operations cannot make all zeros from a nonzero column. Further, observe that the first nonzero column in a row echelon form of $M$ starts with a 1, and all other entries of this column are zero. Therefore, the initial all-zero columns (if any) of $M_1$ and $M_2$, and the first column containing a leading entry in $M_1$ and $M_2$ must be the same (note that $M$ cannot be the zero matrix, since then its row echelon form would also be the zero matrix, so $M$ would not have two different row echelon forms; so the row echelon form of $M$ must have at least one nonzero column).

So, assume that of $M_1$ and $M_2$ agree up to the $n$th column, and the first column that is different in $M_1$ and $M_2$ is the $(n+1)$st column. Then $M_1 \upharpoonright (n+1)$ and $M_2 \upharpoonright (n+1)$ are two different row echelon forms of the matrix $M \upharpoonright (n+1)$. Write $A = M \upharpoonright n$, and let $b$ be the $(n+1)$st column of $M$. Then $(A, b) = M \upharpoonright (n+1)$.

Similarly, write $D = M_1 \upharpoonright n = M_2 \upharpoonright n$, and let $f$ be the $(n+1)$st column of $M_1$ and $g$, the $(n+1)$st column of $M_2$. Then $(D, f) = M_1 \upharpoonright (n+1)$, $(D, g) = M_2 \upharpoonright (n+1)$, and $f \neq g$. As we explained above, we have $n \geq 1$, the initial zero columns of $M_1$ and $M_2$ and the first column containing a leading entry in $M_1$ and $M_2$ must be the same, so $D$ must have at least one leading entry.

Consider the system of linear equations $Ax = b$, where $x$ is an $n \times 1$ matrix (a column vector of length $n$). This system of equations is equivalent to both of the systems $Dx = f$ and $Dx = g$. We will discuss the solvability of the system of equations $Dx = f$ (a similar discussion applies to the system $Dx = g$). Label the columns $D$ containing a leading entry as $l(1)$, $l(2)$, . . ., and label the columns not containing a leading entry as $z(1)$, $z(2)$, . . . . Since, as we mentioned above, $D$ contains at least one leading entry, $l(1)$ is always
defined. As an example, this labeling for a matrix $D$ is shown here:

$$D = \begin{pmatrix}
1 & 2 & 0 & -3 & 2 & 0 & 4 & 0 & 2 \\
0 & 0 & 1 & -2 & 3 & 0 & 3 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix};$$

that is, $l(1) = 1, l(2) = 3, l(3) = 6, l(4) = 8$, and $z(1) = 2, z(2) = 4, z(3) = 5, z(4) = 7, z(5) = 9$. Using this labeling, the solutions of the equation $Dx = f$ can be easily described; however, here we need to know only somewhat less. Namely, we need to know the following: 1) If column of the matrix the matrix does not contain a leading entry, then the equation is solvable. This is easy to see, but the best way to visualize it is to look at a continuation of the above example:

$$(D, f) = \begin{pmatrix}
1 & 2 & 0 & -3 & 2 & 0 & 4 & 0 & 2 & f_1 \\
0 & 0 & 1 & -2 & 3 & 0 & 3 & 0 & 5 & f_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 3 & f_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & f_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

Here we somewhat offset the last column of the matrix to indicate that this column corresponds to the right-hand sides of the equation. This system of equation is unsolvable, since the fifth equation requires $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8 + 0x_9 = 1$, that is, $0 = 1$. On the other hand, if the last column of the matrix the matrix does not contain a leading entry, then the equation is solvable. This is easy to see, but the best way to visualize it is to look at a continuation of the above example:

$$(D, f) = \begin{pmatrix}
1 & 2 & 0 & -3 & 2 & 0 & 4 & 0 & 2 & f_1 \\
0 & 0 & 1 & -2 & 3 & 0 & 3 & 0 & 5 & f_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 3 & f_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & f_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

In this example, the corresponding system of equations can be written as

$$x_1 + 2x_2 - 3x_4 + 2x_5 + 4x_7 + 2x_9 = f_1,$$
$$x_3 - 2x_4 + 3x_5 + 3x_7 + 2x_9 = f_2,$$
$$x_6 + 3x_7 + 3x_9 = f_3,$$
$$x_8 + 4x_9 = f_4,$$

or else as

$$x_{l(1)} + 2x_{z(1)} - 3x_{z(2)} + 2x_{z(3)} + 4x_{z(4)} + 2x_{z(5)} = f_1,$$
$$x_{l(2)} - 2x_{z(2)} + 3x_{z(3)} + 3x_{z(4)} + 2x_{z(5)} = f_2,$$
$$x_{l(3)} + 3x_{z(4)} + 3x_{z(5)} = f_3,$$
$$x_{l(4)} + 4x_{z(5)} = f_4.$$  

In the example, $x_{l(1)} = x_1 = f_1, x_{l(2)} = x_3 = f_2, x_{l(3)} = x_6 = f_3, x_{l(4)} = x_8 = f_4$, and $x_{z(1)} = x_2 = x_{z(2)} = x_4 = x_{z(3)} = x_5 = x_{z(4)} = x_7 = x_{z(5)} = x_9 = 0$ is a solution of the system equations (there are other solutions, but this is of no interest to us here). In general, if $f^T = [f_1, f_2, \ldots],^3$ then a solution of the equation $Dx = f$ is $x_{l(1)} = f_1, x_{l(2)} = f_2, \ldots,$ and $x_{z(1)} = x_{z(2)} = \ldots = 0.$

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^3To save space, we describe a column vector here as the transpose of a row vector, since a row vector is easier to print.
Using this, we can complete the proof as follows. If \( \mathbf{x} \) is a solution of the equation \( A\mathbf{x} = \mathbf{b} \), then \( \mathbf{x} \) is also a solution of the equations \( D\mathbf{x} = \mathbf{f} \) and \( D\mathbf{x} = \mathbf{g} \), and then \( \mathbf{f} = D\mathbf{x} = \mathbf{g} \), showing that \( \mathbf{f} = \mathbf{g} \), contradicting our assumption that \( \mathbf{f} \neq \mathbf{g} \). If the equation \( A\mathbf{x} = \mathbf{b} \) is unsolvable, then the equation \( D\mathbf{x} = \mathbf{f} \) is also unsolvable. In this case the column \( \mathbf{f} \) of \((D, \mathbf{f})\) contains a leading entry in the first row in which the matrix \( A \) contains all zeros. The same argument shows that the column \( \mathbf{g} \) of \((D, \mathbf{g})\) contains a leading entry at the same place. This shows that, \( \mathbf{f} = \mathbf{g} \) again (because the leading entry is 1, and all other entries are 0 in both \( \mathbf{f} \) and \( \mathbf{g} \)). This contradiction completes the proof. \( \square \)