GAME THEORY

Overview

- Game theory explicitly considers interactions between individuals.
- Thus it seems like a suitable framework for studying agent interactions.
- This lecture provides an introduction to some of the concepts of game theory.
- In particular, this lecture considers zero-sum games.

Basic notions

- Being able to figure out how to interact is important.
- Game theory is about games of strategy.
- When one agent makes a move, another agent responds not by chance but by figuring out what is best for it.
- To do this, that agent needs to have some way of knowing what is good for it.
- It also has to have some way of knowing what is good for its opponent (note the adversarial language) in order to try and second guess it.
The basic notions of game theory include:

- players (decision makers);
- choices (feasible actions);
- payoffs (benefits, prizes, rewards . . . ); and
- preferences over payoffs (objectives).

Game theory is concerned with determining when one choice is better than another choice for a particular player.

These “games” can be static or dynamic.

In dynamic games the order of the moves/choices is important.

Here we will only deal with static games.

A simple game is this:

- Player 1 chooses H or T
- Player 2 chooses H or T (not knowing what Player 1 chooses).
- If both choose the same Player 2 wins $1 from Player 1.
- If they are different, Player 1 wins $1 from Player 2.

We can draw this in extensive form.

A strategy for a player is a function which determines which choice he makes at every choice point.

We distinguish games like the one above, in which Player 2 doesn’t know what Player 1 chose, from situations in which Player 2 has perfect information.

The above game is one of perfect information if Player 1 reveals his choice before Player 2 chooses.

The extensive form for this game is on the next slide.
We can also write games in **strategic form**.

Here is the matching game:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>H -1, 1</td>
</tr>
<tr>
<td>Player 2</td>
<td>H -1, 1</td>
</tr>
</tbody>
</table>

The rows are Player 1's moves, the columns are Player 2's moves.

The first payoff in each row is that of Player 1, the second is that of Player 2.

This game is **non-cooperative**

A game is said to be **zero sum** if and only if the payoffs $p_i$ at each terminal of the extensive form are such that:

$$\sum_i p_i = 0$$

---

### Two Person Zero Sum Games

- We can write two person zero sum games in **normal form**
- An example:
  
  $$A = \begin{bmatrix}
-1 & -3 & -3 & -2 \\
0 & 1 & -2 & -1 \\
2 & -2 & 0 & 1
\end{bmatrix}$$
- As with strategic form the rows are the moves of P1 and the columns those of P2
- The entries $a_{ij}$ represent the payoff vector $(a_{ij}, -a_{ij})$.
- How should the players behave?

- One thing that P1 might do is to ask “for each move I might make, what is the worst thing that P2 can do?”.
- Thus he looks for:
  $$\alpha_i = \min_j a_{ij}$$
- He then looks for the move which makes this as good as possible choosing $i^*$ such that:
  $$a_{r,j} = \arg\max_i \min_j a_{ij}$$
- In this case $i^* = \{2, 3\}$.
- Similarly P2 could analyse looking for the move which will minimise his loss given that P1 will try to make this as big as possible choosing $j^*$:
  $$a_{j^*} = \arg\min_j \max_i a_{ij}$$
- In this case, $j^* = \{3\}$. 
Here both agents are trying to do their best to hurt the other since this is the same as profiting as much as they can.

- The value \( \min A \) is called the gain floor of the game.
- The value \( \max A \) is called the loss ceiling of the game.
- Now consider:
  \[
  A = \begin{bmatrix}
  -4 & 0 & 1 \\
  0 & 1 & -3 \\
  -1 & -2 & -1
  \end{bmatrix}
  \]
- \( P_1 \) should take \( i^* = 3 \) and \( P_2 \) should consider \( j^* = 1 \)
- However, if \( P_1 \) knows \( P_2 \) will choose 1, then he should choose 2.
- But if \( P_2 \) knows \( P_1 \) will choose 2, then he should choose 3.
- and so on.

What we have here is an unstable solution.

A solution is stable if no player wants to unilaterally move away from the solution.

A solution is inadmissible if there are solutions that produce better payoffs for all players than the given solution.

What we want is a way of identifying stable solutions.

- It is easy to see that both players will settle on \((i^*, j^*)\) if \( \min A = \max A \).
- In this case:
  \[
  \min A r_j = \min A = \max A = \max A
  \]

If \( \min A = \max A \) then:
- \( A \) has a saddle point
- The value for the game is \( V = \min A = \max A \)

This works fine for games which do have a saddle point, however, what happens if:

\[
\min A < \max A
\]

as in the game:

\[
A = \begin{bmatrix}
3 & -1 \\
0 & 1
\end{bmatrix}
\]

- Here \( P_1 \) has \( \min A = 0 \) and \( i^* = 2 \).
- For \( P_2 \), \( \max A = 1 \) and \( j^* = 2 \)

What we want is a "spy-proof" strategy.

This is one which works even if the other player knows what the strategy is.

We manage this by moving from a pure strategy in which a player makes a definite choice of move...

...to a mixed strategy in which a player makes a random choice across a set of pure strategies.

### Mixed Strategies
More formally, P1 picks a vector of probabilities:

\[ x = (x_1, x_2) \]

where

\[ \sum_i x_i = 1 \]

and

\[ x_i \geq 0 \]

P1 then picks strategy \( i \) with probability \( x_i \).

To determine the strategy, P1 needs then to compute the best values of \( x_i \) and \( x_j \).

These will be the values which give P1 the highest expected payoff for his mixed strategy.

P1’s analysis would be something like this:

Now, let’s consider the payoff’s the players will expect.

With P1 having mixed strategy \( (x_1, x_2) \) and P2 having \( (y_1, y_2) \), the value of the game will be:

\[
V = 3x_1y_1 + 0(1 - x_1)y_1 - x_2(1 - y_1) + (1 - x_1)(1 - y_1) = 3x_1y_1 - y_1 - 2x_1 + 1
\]

Now, let’s assume that P1 uses \( x_1^* = 0.2 \) as calculated above. Then:

\[
V = 5(0.2y_1) - y_1 - 2(0.2) + 1 = 0.6
\]
Similarly, if $P2$ picks $y^*_1 = 0.4$ then:

$$V = 0.6$$

- The neat thing is that the expected value for one player does not depend upon the strategy of the other player.
- This result generalises.
- Von Neumann’s Minimax Theorem shows that you can always find a pair of mixed strategies $x^*$ and $y^*$ which result in $P1$ and $P2$ have the same expected value for the game.

This is important because it means we have something similar to:

$$[(A) = [(A)]$$

- In other words, there is a kind of stability.
- It is also possible to prove that either player can do no better using a pure strategy than he can using a mixed strategy.
- This makes it possible for one player to know that the other player is going to use a mixed strategy.
- This is the key to stability.

**Summary**

- This lecture has introduced some of the basic ideas of game theory;
- It has covered the notion of a stable solution to a game; and
- It has covered pure strategy and mixed strategy solutions.