On the Interaction between Computer Science and Economics

Rohit Parikh
Eric Pacuit

January 21, 2004

Abstract

1 Introduction

In [**?**], Parikh asks "is it possible to create a theory of how social procedures works with a view to creating better ones?" This paper will survey some of the mathematical tools that have been developed over the years that may help answer this question. But what exactly is meant by a *social procedure*?

Let's start with an example. Suppose that Ann would like Bob to attend her talk; however, she only wants Bob to attend if he is interested in the subject of her talk, not because he is just being polite. There is a very simple procedure to solve Ann's problem: have a (trusted) friend tell Bob the time and subject of her talk. Taking a cue from computer science, we can ask is this procedure correct? Just as we can show that Quick sort correctly sorts an array, perhaps we can show that this simple procedure correctly solves Ann's problem. While a correct solution for the Quick sort algorihtm is easily defined, it is not so clear how to define a correct solution to Ann's problem. If Bob is actually present during Ann's talk, can we conclude that Ann's procedure succeeded? Of course not, Bob may have figured out that Ann wanted him to attend, and so is there only out of politeness. A correct description of a solution to Ann's problem must describe the knowledge that each agent has about the situation at hand. In other words, we will say that Ann's procedure succeeded if it induces the following situation:

1. Ann knows about the talk.
2. Bob knows about the talk.
3. Ann knows that Bob knows about the talk.
4. Bob does not know that Ann knows that he knows about the talk.

5. *And nothing else.*

Note that the last point is important, since otherwise Bob may feel social pressure to attend the talk.

In computer science it is often convenient to view a computational procedure, or a program, as a relation on a set of states, where a state can be thought of as a function that assigns a value to every possible variable and a truth value to all propositions. We can continue the analogy with computational procedures, by assuming that a social procedure is a relation on states, where each state is intended to model the level of knowledge and belief among a group of agents. Thus in order to develop a realistic model of social procedures, we need a realistic model of multi-agent knowledge and belief. The following example taken from [**?**] will show that this is not quite enough – we need a game theoretic model in which utilities can be represented.

Suppose that Ann and Bob each own a horse and each would like to sell their horse. Also, suppose that Charles is willing to pay $100 to the owner whose horse can run the fastest. In fact, Ann's horse is faster running at 30 MPH, while Bob's horse can only run at 25 MPH. The obvious procedure that Charles can use to determine who to give the money to is to ask Ann and Bob to race, and give $100 to the winner. Why is this procedure successful? Well, Charles has created a situation in which both Ann and Bob have an incentive to make their horse run as fast as possible. The situation can be represented by a strategic game, a detailed discussion can be found in [**?**].

On the other hand, if Charles wants to buy the *slower* horse, the above procedure will no longer work. Since Ann and Bob both would prefer to sell their horse than not to sell their horse, they both have an incentive to make their horse run as *slow* as possible. This will create the strange situation in which race will never actually have a chance to begin. Note that if Charles deceives Ann and Bob into thinking that he intends to give $100 to the faster horse, he can in fact determine which horse is slower. However, we certainly do not want a theory of social procedures that promotes deception. Is there a procedure that Charles can follow to determine which horse is slower without deceiving Ann and Bob? The answer as discussed in [**?**] turns out to be yes: Charles can ask Ann and Bob to race, but they should switch horse. Since Ann and Bob are racing on each other's horse, the situation is simlar to the original race in which both have an incentive to make the horse they are riding run as fast as possible.

Note that if we did not take into account the utilities of each player, the first race (without the agents switching horses) would seem to work in both cases. In the first case Charles would give $100 to the winner and in the second case $100 to the loser. The need for a different procedure only becomes apparent when we notice that the incentive for Ann and Bob to push their horse to run as fast as possible is necessary in order for the race to suceed.

These two examples demonstrate that in order to develop a theory of social procedures we must be able to represent an agent's knowledge and beliefs *and* an agent's utilites. But which takes precedence? Do agents act according to ther utilities or according to the beliefs they have about the situation at hand? Consider a traffic light, which can be understood as a piece of social software. The traffic light was designed to stop accidents at a busy intersection. When you come upon a red light, do you stop because it is against the law to proceed (i.e. you assign a greater utility to stopping than going), or because you know that the other driver's believe that you will stop? Actually, you probably do not think much about it, and just stop because that is what you did at the previous intersection. Nonetheless, the traffic light does work, and we would like a mathematical model that explains why.

2 Models for Individual Knowledge and Beliefs

we will beging by looking at some mathematical models that can describe individual knowledge and beliefs of agents. We will present some standard mathematical models which offer a static model of a situation. These mathematical models have shown up in such fields as Computer Science, Philosophy, Game Theory, and Economics to name a few. This section is not intended to be a complete survey of the field, and so there may be some important concepts which are glossed over or even ignored. Refer to [**?**, **?**, **?**] and the text [**?**] for an more information about formal epistemic logic. [**?**] has a nice overview of the semantic and probabilistic models of knowledge and belief.

Suppose that we fix a social situation involving the agents $\mathcal{A}$. How should we represent the knowledge of each agent? A natural assumption is that no agent can have *all* the information about a situation. For one thing agents are computationally limited and can only process a bounded amount of information. Thus if a social situation can only be described using more bits of information than an agent can process, then an agent can only maintain a portion of the total information describing the situation. Another point justifying this assumption is that the observational power of an agent is limited. For example suppose that the exact size of a piece of wood is the only relevant piece of information about some situation. While any agent will have enough memory to maintain this single piece of information, any measuring device is subject to error. Therefore it is fair to assume that any two agents may have different views or interpretations of the same situation. It is this uncertainty about a situation that is captured by the formal models in the following sections.

2.1 Kripke Structures

We first look a formal epistemic logic. Formal epistemic logic has its roots in Hintikka's *Knowledge and Belief.* In formal epistemic logic, a formal language is defined that can express statements such as "agent i knows P" or "agent i knows that agent j does not know that it is raining". Then a set of axiom schemes and rules are offered which are assumed to describe the reasoning power of the agents involved. Using the defined deductive system plus perhaps some non-logical axioms relevant to a particular situation, all relevant facts can be derived. The final piece is a semantics in which the formal language can be interpretted. It is in the semantics that we find the model of agent uncertainty. The intuition is that an agent in a particular situation, called a possible world or state, considers other situations possible. Thus, we can say that an agent "knows" a fact if that fact is true in all possible worlds that the agent considers possible. The word "knows" is in quotes, since the intensional notion being modelled is dependant on the properties of the model.

Formal epistemic logic is an application of modal logic. Many important definitions and theorems are stated in Appendix A. We will give a brief overview of the formal language and some important logics. Let Φ_0 be a countable set of propositional variables. Let $\mathcal{F}_{\Phi_0,PC}$ be the set of well-formed formulas built from the standard propositional connectives, say $\neg$ and $\wedge$, using propositional variables from the set Φ_0. if Φ_0 is understood or not important, we may write $\mathcal{F}(PC)$. We take the other standard connectives $(\vee, \rightarrow, \leftrightarrow)$ as being defined as usual. A multi-agent epistemic language adds an operator K_i for each agent $i \in \mathcal{A}$. If ϕ is a formula, then the intended interpretation of $K_i\phi$ is that "agent i knows ϕ". Let $\mathcal{L}_{PC}(\Phi_0, K, \mathcal{A})$ be the set of well-formed formulas in this extended language. We will make use of the following notation throughout this survey, $\mathcal{L}_{PC}(\Phi_0, O^1, O^2, \ldots, O^n, \mathcal{A})$ is the set of well-formed formulas built from formulas of propositional calculus and closed under each operator O_i^j, $j = 1, \ldots, n$ for each agent $i \in \mathcal{A}$.

The approach of formal epistemic logic is to propose a certain set of axiom schemes, then using these axiom schemes (plus some rules of inference) one can deduce the formulas an agent knows (or believes). The following is a list of some common axioms. We will use $\Box_i$ to stand for either K_i (agent i knows...) or B_i (agent i believes ...). Let $\phi, \psi \in \mathcal{L}_{PC}(\Box, \mathcal{A})$ be arbitrary formulas,

K	$\Box_i(\phi \rightarrow \psi) \rightarrow (\Box_i\phi \rightarrow \Box_i\psi)$	*Distributivity*
T	$\Box_i\phi \rightarrow \phi$	*Truth*
4	$\Box_i\phi \rightarrow \Box_i\Box_i\phi$	*Positive introspection*
5	$\neg\Box_i\phi \rightarrow \Box_i\neg\Box_i\phi$	*Negative introspection*
D	$\neg\Box_i\bot$	*Consistency*

Since ϕ and ψ are arbitrary formulas, the above are axiom schemes. We can then define a logic $\Lambda(A_1, \ldots, A_n, R_1, \ldots, R_m)$ over some language $\mathcal{L}$ to be the set of formulas that can be deduced from axioms $A_1, \ldots, A_n$ and rules $R_1, \ldots R_m$. If

Λ is a logic over a language $\mathcal{F}$, then we often write $\vdash \phi$ if $\phi \in \Lambda$. The most common rules of inference are modus ponens (MP) and necessitation (N), i.e. from ϕ infer $\Box_i \phi$ for each $i \in \mathcal{A}$. Since propositional modal logic is an extension of propositional calculus, each logic Λ over $\mathcal{L}$ will contain all the propositional tautologies in the language of $\mathcal{L}$. Let PC be a[1] collection of axiom schemes of propositional calculus. We define

$$\begin{aligned}
S5 &= \Lambda(PC, K, T, 4, 5, MP, N) \\
KD45 &= \Lambda(PC, K, D, 4, 5, MP, N) \\
S4 &= \Lambda(PC, K, T, 4, MP, N) \\
T &= \Lambda(PC, K, T, MP, N) \\
K &= \Lambda(PC, K, MP, N)
\end{aligned}$$

Any logic containing the K axiom and the rule N, is called a **normal modal** logic. Suppose that $\mathcal{L}_K(\mathcal{A}) = \mathcal{L}_{PC}(K, \mathcal{A})$ and $\mathcal{L}_B(\mathcal{A}) = \mathcal{L}_{PC}(B, \mathcal{A})$ and $\mathcal{L}_{K,B}(\mathcal{A}) = \mathcal{L}_{PC}(K, B, \mathcal{A})$. In the interest of brevity, we may write $\mathcal{L}_K, \mathcal{L}_B$, or $\mathcal{L}_{K,B}$ if the $\mathcal{A}$ is understood. The arguments for and against the above axioms are well known and will not be discussed here. Additional information can be found in [**?**, **?**]. Instead we turn to semantics.

Let W be a set of states, or worlds. Intuitively a state, or world, $w \in W$ contains all the relevant facts about a particular situation. The uncertainty of the agents is represented by a relation $R_i \subseteq W \times W$ for each agent $i \in \mathcal{A}$. Intuitively wR_iv means that from state w, agent i thinks that state v is the actual situation.

Definition 1 (Kripke Frame) *Given a finite set of agents $\mathcal{A}$, the pair*[2] $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ *is called a* **Kripke frame**. *Where W is any set, and $R_i \subseteq W \times W$ for each $i \in \mathcal{A}$.*

A Kripke frame is a very abstract mathematical model of uncertainty, and so we need some way to connect this abstract model to actual situations. This connection is provided in the next two defintions.

Definition 2 (Kripke Model) *Given a frame $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ and a set of propositional variables Φ_0. A* **Kripke model** *based on $\mathcal{F}$ is a triple $\langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$, where $V : \Phi_0 \to 2^W$.*

We can now define truth

Definition 3 (Truth in a Kripke model) *Given a language $\mathcal{F}(\Phi_0, \Box, \mathcal{A})$ and a model $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$. We define a relation $\models \subseteq W \times \mathcal{F}(\Phi_0, \Box, \mathcal{A})$ as follows. Suppose that $s \in W$, $i \in \mathcal{A}$, $P \in \Phi_0$ and $\phi, \psi \in \mathcal{F}(\Phi_0, \Box, \mathcal{A})$, note that we write $\mathcal{M}, s \models \phi$ instead of $(s, \phi) \in \models$*

[1] There are many different axiomatizations of propositional calculus, choose your favorite.

[2] Notice that we do not put the set of agents as an explicit parameter in the definition of a Kripke frame. Unless otherwise stated we assume that we have a finite fixed set of agents. If confusion may arise, we will include the set of agents as a third parameter.

1. $\mathcal{M}, s \models P$ *if* $P \in V(s)$

2. $\mathcal{M}, s \models \phi \vee \psi$ *if* $\mathcal{M}, s \models \phi$ *or* $\mathcal{M}, s \models \psi$

3. $\mathcal{M}, s \models \neg\phi$ *if* $\mathcal{M}, s \not\models \phi$

4. $\mathcal{M}, s \models \Box_i \phi$ *if for each* $v \in W$*, if* wRv*, then* $\mathcal{M}, v \models \phi$

We define $\Diamond_i \phi \stackrel{\text{def}}{=} \neg\Box_i\neg\phi$. If the model $\mathcal{M}$ is understood we may write $s \models \phi$. $\mathcal{M}, s \models \phi$ for all states $s \in W$, then we say that ϕ is valid in $\mathcal{M}$ and write $\mathcal{M} \models \phi$. If ϕ is valid in all models based on a frame $\mathfrak{F}$, then we say that ϕ is valid in the frame and write $\models \phi$. We will make use of the following notation, let $s \in W$ and ϕ any formulas,

$$\begin{array}{rcl} R(s) & := & \{t \mid sRt\} \\ R_\phi(s) & := & \{t \mid sRt \text{ and } t \models \phi\} \end{array}$$

Given a state s, then $R(s)$ represents the set of states that an agent considers possible from state s. Call a frame $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ reflexive if R is reflexive. It is well known that $\Box_i\phi \to \phi$ is valid in a frame $\mathcal{F}$ iff $\mathcal{F}$ is reflexive. More information on this can be found in the appendix and in [**?**]. We only give the correspndances between the axioms and properties of the frame. Let $\Box$ stand for either K_i or B_i.

Axiom is valid in the frame	Correpsonding property of the frame
$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	(valid in all frames)
$\Box\phi \to \phi$	Reflexivity
$\Box\phi \to \Box\Box\phi$	Transitivity
$\neg\Box\phi \to \Box\neg\Box\phi$	Euclidean
$\neg\Box\bot$	Serial

The general resul about the frame definable formulas and their correspondance to modal axioms can be found in [**?**]. We will focus on the logics $S5$, $KD45$ and $S4$ since they are the most relevant to our original goal of modeleing multi-agent beliefs and knowledge.

We will call a frame $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ reflexive if R_i is reflexive for each $i \in \mathcal{A}$. Similarly, for the above properties. We will now state many of the fundamental of formal epistemic logic. These results are proved and can be found in [**?**]

Theorem 4 $S5$ *is sound and strongly complete with respect to the class of all frame* $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ *where* R *is an equivalence relation.*

Theorem 5 $S4$ *is sound and strongly complete with respect to the class of all frame* $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ *where* R *is reflexive and transitive.*

Theorem 6 $KD45$ *is sound and strongly complete with respect to the class of all frame* $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ *where* R *is serial, transitive and euclidean.*

All defintions and general completeness and soundess results can be found in [**?**]. Given the soundness and completeness proofs, we may call any model based on a frame in which the relation is an equivalence relation an $S5$ model. The proof of completeness uses a canonical model construction, which will be discussed in Section 2.5.

Given any formula ϕ of a particular question, we can ask is ϕ satisfiable (does it have a model) or is it valid (true in all models). In light of completeness this is the same question as does ϕ have a proof. Since the logics we study enjoy the finite model property, it is easy to see that the satisfiability problem is decidable. Ladner showed that the satisfiability problem for $S5$ (single-agent case) is NP-complete, while the same problem for K, T and $S4$ is $PSPACE$-complete. It can also be shown that the satisfiablility problem for single-agent $KD45$ is NP-complete. For the multi-agent case, the satisfiability problem for each of these logics is $PSPACE$-complete. [**?**] has an excellent discussion about complexity and modal logic.

The essential facts needed to show that the satisfiability problem of $S5$ and $KD45$ is NP-complete (as opposed to $PSPACE$-complete) are the following two theorems as stated in [**?**]

Proposition 7 *An $S5$ formula ϕ is satisfiable if and only if it is satisfiable in a single-agent Kripke structure where the relation is an equivalence relation and there are at most $|\phi|$ states.*

Proposition 8 *An $KD45$ formula ϕ is satisfiable if and only if it is satisfiable in a single-agent Kripke structure where the relation is Euclidean, transitive and serial and there are at most $|\phi|$ states.*

2.2 Aumann Structures

Suppose that the group of agents $\mathcal{A} = \{1, \dots, n\}$ are involved in a social situation. We will assume that the situation is strategic in the sense that the agents are trying to achieve conflicting outcomes. An important problem is how to explain *why* the agents make the choices that they do. In other words, what induces an agent to choose a particular strategy. A basic assumption is that agents attach utilities to every conceivable outcome. The agent will then choose an action so as to maximize his or her expected utility. By a strategy profile, we mean a set of strategies, one for each agent[3]. Suppose that σ is a strategy profile. A natural question to ask is is σ a stable profile? In other words, if we fix an agent $i \in \mathcal{A}$ and suppose that all agents other that i (usually denoted by $-i$) will follow the strategy prescribed by σ, should i follow the strategy prescribed by σ? If yes, then that particular choice is a **best-response** for agent i.

[3] formally, we take advantage of the fact that we can assume an arbitrary linear order on the agents, so a strategy is just a sequence of individual strategies

If this is true for all agents, then the strategy profile is said to be in **Nash** equilibrium. More discussion of game theory can be found in [**?**] and in Appendix B.

Lurking in the background of the above discussion are assumptions about what agents believe about the situation, the other players beliefs about the situation, the other players belief about the agent's beliefs, and so on. It is these assumptions that many of the authors are trying to make explicit.

One of the first attempts to formalize these assumptions was by Aumann [**?**]. We follow the discussion in [**?**]. The intuition behind knowledge spaces is as follows. We want to model the knowledge an agent has about a certain situation (including the knowledge that an agent has about other agents). Define a state of nature to be a complete run of a game[4]. In other words, a state can be thought of as a set of propositions that are true of a given run of the game. Now given a particular run of the gamee, an agent may be uncertain about a particular proposition, say ϕ. In this case the agent can imagine that a situation in which ϕ is true and one in which ϕ is false. Notice that here we actually have *three* possiblities. The actual world in which either ϕ is true or it is false, the world in which the agent thinks ϕ is true and the world in which the agent thinks ϕ is false.

Let W be a set of worlds. In the previous section we defined a formal language that can express various knowledge-theoretic statements about the agents, and interpretted this language in a Kripke model. In this section we will reason semantically, no object language is defined. Instead we first make a distinction between a state of nature and states of the world. Let S be the set of all states of nature. A state of nature is a complete description of the exogenous parameters that do not depend on the players' uncertainties.

Definition 9 (Aumann Frame) *Given a finite set of agents $\mathcal{A}$, the pair $\langle W, \{P_i\}_{i\in\mathcal{A}}\rangle$ is called an* **Aumann frame***, where W is any set, and for each $i \in \mathcal{A}$, $P_i : W \to 2^W$ is a function.*

There is an obvious translation between Kripke frames (defintion 1) and Aumann frames. This connection is discussed below. We need give the analogous defintion of a model.

Definition 10 (Aumann Model) *Given a Aumann frame $\mathcal{F} = \langle W, \{P_i\}_{i\in\mathcal{A}}\rangle$ and a set of states S, an* **Aumann model based on S** *is a triple $\langle W, \{P_i\}_{i\in\mathcal{A}}, \sigma\rangle$, where $\sigma : W \to S$.*

So, σ is analogous to a valuation function, it assigns to each world a state of nature in which every proposition *not about the uncertainty of the agents* is either true or false.

Intuitively, we say that event $E \subseteq W$ is true at state w if $w \in E$. Thus we are making essential use of the fact that we can identify a proposition with the

[4]We will use game and strategic situation interchangeably

set of worlds in which it is true. In the previous section we definied an object language that could express statements of the form "agent i knows ϕ", and interpretted these formulas in a Kripke model. In this section we have no such object language. Reasoning about agents is done purely semantically. We still need to be able to express statements abut agents' knowledge. In this framework we will think of notions like knowledge and beliefs as set-valued operators that map propsotions (subsets of W) to propositions. Given any possibility function $P : W \to 2^W$, we can associate a possibility operator $\mathsf{P} : 2^W \to 2^W$ defined by

$$\mathsf{P}(E) = \{w \mid P(w) \subseteq E\}$$

for any subset $E \subseteq W$.

Given an arbitrary set operator $\mathsf{P} : 2^W \to 2^W$, the following properties can be assumed of P. Let E, F be any two subsets W

P1 $\mathsf{P}(E) \cap \mathsf{P}(F) = \mathsf{P}(E \cap F)$

P2 $\cap_{j\in J}\mathsf{K}(E_j) = \mathsf{P}(\cap_{j\in J}E_j)$, for any index set J[5]

P3 $\mathsf{P}(E) \subseteq E$

P4 $\mathsf{P}(E) \subseteq \mathsf{P}(\mathsf{P}(E))$

P5 $\overline{\mathsf{P}(E)} \subseteq \mathsf{P}(\overline{\mathsf{P}(E)})$

P6 $\mathsf{P}(E) \subseteq \overline{\mathsf{P}(\overline{E})}$

where $\overline{E}$ means set complement with respect to W. We first note that $P3, P4, P5$ and $P6$ are the obvious analogues[6] to axioms $T, 4$, 5 and D respectively. The property $P1$ corresponds to the K axiom[7]. However, notice that there is no analogue to MP, PC and N in the above axioms. Obviously $P2$ implies $P1$, but notice that contrary to $P1$ there is no formal analogue to $P2$, since the index set J can be infinite.

In [**?**], Halpern offers a proof that $P2$ follows from $P1, P3, P4$ and $P5$; and in fact, if any of the properties are weakened, the proof fails. Thus, this is an example of a difference between $S5$ and $S4$ that cannot be expressed syntactically. Obviously, if we consider a formal language that allows infinite conjunctions and disjunctions, this distinction can be expressed[8].

Let $P : W \to 2^W$ be any function. We define the following properties of P:

[5]When $J = \varnothing$, we get $\mathsf{K}(\Omega) = \Omega$

[6]under the appropriate translations, where propositions are replaced by events, $\vee$ by $\cup$, $\wedge$ by $\cup$ and $\neg$ by set complement

[7]It is easy to see that the K axiom is equivalent to $K(\phi \wedge \psi) \leftrightarrow (K\phi \wedge K\psi)$, in the presence of PC and MP

[8]In fact, Halpern goes on to show that there are other distinctions in formal conditional logic that cannot be expressed syntactically only involving finite conjunctions and disjunctions

Reflexive $\forall w \in W,\, w \in P(w)$

Transitive $\forall w, v \in W,\, v \in P(w) \Rightarrow P(v) \subseteq P(w)$

Euclidean $\forall w, v \in W,\, v \in P(w) \Rightarrow P(w) \subseteq P(v)$

Serial $\forall w \in W,\, P(w) \neq \varnothing$

We will call $\langle W, \{P_i\}_{i\in\mathcal{A}}, \sigma\rangle$ a **knowledge space** if P is reflexive, transitive and Euclidean[9]. We will call $\langle W, \{P_i\}_{i\in\mathcal{A}}, \sigma\rangle$ a **belief space** if P is serial, transitive and Euclidean. The above properties of a possibility operator correspond to properties on the possibility funciton of the corresponding frame. In fact we can state a theorem analogous to the soundness and completeness theorem of the previous section. The following theorem is easy to prove and can be thought of as a soundess theorem.

Theorem 11 (Halpern [?]) *Let $\mathcal{F} = \langle W, P\rangle$ be an Aumann frame[10]. Let $\mathsf{P} : 2^W \to 2^W$ be defined from P as above. Then P satisfies $P2$ (and hence $P1$). Also we have the following correspondance: if P is reflexive, the P satisfies $P3$, if P is transitive, then P satisfies $P4$, if P is Euclidean, then P satisfies $P5$ and if P is serial, then P satisfies $P6$.*

Theorem 12 (Halpern [?]) *Suppose that P is any operator satisfying $P2$, then there is a frame $\langle W, P\rangle$ such that the operator defined from P is exactly P. Moreovere, if P satisfies $P3$, then P is reflexive, if P satisfies $P4$, then P is transitive, if P satisifies $P5$, then P is Euclidean, and if P satisfies $P6$, then P is serial.*

This completeness proof is much easier than the canonical construction needed in the previous section. We will not give aproof here, but one can be found in [?] and [?]. We only remark that given a possibility operator $\mathsf{P} : 2^W \to 2^W$, we define the possibility function P as follows:

$$P(w) = \cap\{E \mid w \in \mathsf{P}(E)\}$$

It is straightforward to checke that P saisfies the appropriate properties.

2.3 History Strucutures

3 Levels of Knowledge

Recall the discussion of example 1. Let P be the proposition "Ann's talk is at 1 PM in room 4435". Let $K_A P$ mean that Ann knows P and $K_B P$ mean that Bob knows P. Then Ann wants to induce the following situation:

[9] and hence symmetric. It is easy to show that P is symmetric. Let $v \in P(w)$, then by Transitivity and Euclideaness, $P(w) = P(v)$. And so by reflexivity, $w \in P(v)$.

[10] Without loss of generality, we will only consider the single agent case, and so the subscripts will be left out

1. $K_A P$
2. $K_B P$
3. $K_A K_B P$
4. $\neg K_B K_A K_B P$

We will say that the level of knowledge of the proposition P is the set $\{K_A, K_B, K_A K_B\}$. This suggests a formal language theoretic framework for talking about group knowledge. This framework was first discussed in [**?**] and later in [**?**]. The idea is to consider a finite alphabet of knowledge operators (one for each agent) and define a level of knowledge of a proposition to be a set of strings over the alphabet. We will first discuss the problem in general, taking the alphabet to be any finite set of strings and a level to be any set of strings.

3.1 Basic Results

Let Σ be any finite alphabet[11]. We will call any set $L \subseteq \Sigma^*$ a **level**. In the next section will discuss levels of knowledge, levels of beliefs or a mixture of the two, but for now we will develop just a general theory of levels over some finite alphabet. Formally a level is any set of strings, i.e.a langugae. Let Σ^* be the set of finite strings over Σ, Σ^ω the set of infinite strings over Σ, and $\Sigma^{\leq\omega} = \Sigma^* \cup \Sigma^\omega$ be the set of all strings (finite or infinite).

Definition 13 (Simple String) *Given any string $x \in \Sigma^*$. We call x* **simple** *if x does contain any repeated letters. Let $\Sigma^s \subseteq \Sigma^*$ be the set of all simple strings.*

For example if $\Sigma = \{a, b, c\}$, then the string *abacb* is simple but *abbca* is not simple. Notice that we have defined Σ^s to be the set of *finite* simple strings. We similarly could define $\Sigma^s \subseteq \Sigma^\omega$ or $\Sigma^s \subseteq \Sigma^{\leq\omega}$. Unless otherwise noted, we will assume that $\Sigma^s \subseteq \Sigma^*$. We will say that a level L is **simple** if $L \subseteq \Sigma^s$.

There is a natural embeddability ordering on the set of strings over Σ which turns out to be important for levels of knowledge.

Definition 14 (Embedibility Ordering) *Given any two strings $x, y \in \Sigma^{\leq\omega}$, we say x is* **embeddable** *in y, written $x \leq y$, if all symbols of x occurr in y in the same order but not necessarily consecitively. Formally, we can define $\leq$ as follows:*

1. *$x \leq x$ and $\epsilon \leq x$ for all $x \in \Sigma^{\leq\omega}$*
2. *$x \leq y$ if there exists $x', x'', y', y'', (y, y'' \neq \epsilon)$ such that $x = x'x''$, $y = y'y''$ and $x' \leq y'$, $x'' \leq y''$.*

[11] We will often think of each element of Σ as a modal operator, such as knowledge (K_i), belief (B_i) or common knolwedge (C_G).

$\leq$ *is the smallest relation satisfying (1) and (2).*

For example *abbc* is embeddable in *accbbc*, itself, and *aaabbbbbcb* but not in *aaacc*. We will formally show $abbc \leq accbbc$. In part 2 of the definition above, take $x = abbc, y = accbbc$ and define $x' = a, x'' = bbc, y' = a$, and $y'' = ccbbc$. Then $x' \leq y'$ by 1 above. To se $x'' \leq y''$, let $z' = \epsilon, z'' = bbc, w' = cc, w'' = bbc$. Then $x'' = z'z''$ and $y'' = w'w''$, $z' \leq w'$ by 1, and $z'' \leq w''$ also by 1.

Recall that a relation is called well-founded if every nonempty subset has a minimal element; and a relation is called a well-partial order if it is well-founded and every linear order that extends it is a well order. Equivalently a relation is a well-partial order if it is well founded and every set of mutually incomparable elements is finite. The following fact was shown by Graham Higman. See [**?**] for details.

Fact 1 (Higman) $\leq$ *is a well partial order.*

The following facts are straightforward.

Fact 2 *Embedibility can be tested in linear time by a two tape finite automaton*

Fact 3 *For any string x there is a shortest simple string y such that $y \leq x$. Define $Sim(x) = y$.*

Given any level L, define $L^s = Sim(L^*)$.

The following notion is standard given an order ($\leq$) on the set of strings.

Definition 15 (Downward Closed) *A* downward closed *subset of $\Sigma^{\leq \omega}$ is a subset X such that if $x \in X$ and $y \leq x$, then $y \in X$. The* downward closure *of a set X is $dc(X) = \{y \in \Sigma^{\leq \omega} \mid y \leq x \text{ for some } x \in X\ \}$.*

The notion of upward closed can be defined in a similar way. The following properties of downward closed sets can be found in [**?**]. The proofs are straightforward applications of the definitions and so will be left to the reader.

Fact 4 *If Y is downward closed, then for each X, $X \subseteq Y$ iff for all $x \in X$, there is a $y \in Y$ such that $x \leq y$.*

Fact 5 $dc(X \cup Y) = dc(X) \cup dc(Y)$

Fact 6 $dc(XY) = Sim(dc(X)dc(Y))$*, where XY is the conatenation of X and Y, that is $XY = \{w \mid w = xy, x \in X, y \in Y\}$; and $Sim(X) = \{Sim(x) \mid x \in X\}$.*

Since $\leq$ is a well-partial order, any set of incomparable elements is finite. In particular if L is a level, then the set of $\leq$-minimal elements of L and $\leq$-maximal elements of L are both finite. If L is downward closed, then the set of minimal elements is the set of all elements of Σ that appear in L. So, obviously two different downward closed sets may have the same minimal elements. Two different downward closed sets may also have the same maximal elements. For example if $L = \{a, b\}^*$ and $L' = \{c, d\}^*$, then both L and L' have no maximal elements. However, we can characterize any downward closed simple level by the minimal elements of $\overline{L}$ (the complement of L)..

Theorem 16 (Parikh, Krasucki) *Let Σ be any finite set. There are only countably many downward closed simple levels, and each is a regular subset of Σ^*.*

Proof Let $L \subseteq \Sigma^*$ be downward closed set and simple. Then $\overline{L}$ is upward closed. Let $m(\overline{L})$ be the set of minimal elements of $\overline{L}$. Since the minimal elements must be incomparable and $\leq$ is a well-partial order, $m(\overline{L})$ is finite. Suppose that $m(\overline{L}) = \{x_1, x_2, \ldots, x_n\}$. Then

$$L = \{y \mid \forall x \in m(\overline{L}), x \not\leq y\}$$

Since $m(\overline{L})$ is finite, a finite automonton can clearly be designed to test whether $x \leq y$ for some input y. Hence, L is a regular subset of Σ^*. The fact that there are only countably many levels follows immediately.

■

If L is finite set of simple strings, then we can use the maximal elements to determine whehter L is downward closed.

Theorem 17 ([?]) *If L is a non-empty finite subset of Σ^s, then L is downward closed iff there is a set of simple strings $\{x_1, x_2, \ldots, x_k\}$ such that,*

$$L = \cup_{i=1}^{k} dc(\{x_i\})$$

In fact, we may take the set $\{x_1, \ldots, x_k\}$ to be the maximal elements of L.

Proof By fact **??**, $\cup_{i=1}^{k} dc(\{x_i\})$ is downward closed. Suppose that $L = \{x_1, x_2, \ldots, x_n\}$ is downward closed. Let $L' = \cup_{i=1}^{n} dc(\{x_i\})$. Clearly $L \subseteq L'$. Let $y \in L'$. Then $y \in dc(\{x_i\})$ for some i. Suppose that $y \notin L$. But then there is an $x_i \in L$ and $y \leq x_i$ but $y \notin L$. This contradicts the fact that L is downward closed. Hence $y \in L$ and so $L = L'$ as desired. The last fact is obvious.

■

The above theorem shows that every finite downward closed set is the finite union of the downward closure of the maximal elements. What about infinite downward closed sets? The following notion of star-linear sets can be used to characterize all infinite downward closed sets.

Definition 18 *A subset L of Σ^s is* star-linear *iff there exist strings $x_1, \ldots, x_{m+1}$ and subsets $\Delta_1, \ldots, \Delta_m$ of Σ such that*

$$L = dc[(\{x_1\})\Delta_1^*(\{x_2\})\Delta_2^* \cdots \Delta_m^*(\{x_{m+1}\})] \cap \Sigma^s$$

The following theorem whose proof can be found in [?] completely characterizes downward closed sets.

Theorem 19 ([?]) *If L is downward closed, then L has a unique representation as a finite minimal union of star-linear sets.*

3.2 Levels of Knowledge and Beliefs

In the previous section the alphabet Σ was any abritrary finite set. However, in what follows we will assume that the elements of Σ are modal operators. Let $\mathcal{A} = \{1, 2, \ldots, n\}$ be a set of agents. Then the **modal alphabet** based on $\mathcal{A}$ is the set $\Sigma_{\mathcal{A}} = \{\Box_1, \ldots, \Box_n\}$. We will write Σ for $\Sigma_{\mathcal{A}}$ when the set of agents is understood.

Let $\mathcal{M} = \langle W, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ be any Kripke model, $w \in W$ and ϕ be any formula. For now we will assume that ϕ is modal-free, i.e. a formula of propositional logic. Then define

$$L(w, \phi) = \{x \mid x \in \Sigma^*_{\mathcal{A}}, \mathcal{M}, w \models x\phi\}$$

We say $L(w, \phi)$ is the level of ϕ at state w. A natural question is what types of sets can arise as levels of some formula in a Kripke model? The answer to this questions depends on the underlying logic, i.e. what kind of operator $\Box_i$ is. We, of course, assume that each modal operator is normal.

The following fact is easy to check.

Fact 7 *Let $\Sigma = \{\Box_1, \ldots, \Box_n\}$. If for all formulas ϕ and all strings $x, y \in \Sigma^*$, $a \in \Sigma$ we have*

$$\models xay\phi \leftrightarrow xaay\phi$$

Then for all Kripke models and states w, $xay \in L(w, \phi)$ iff for all $j \geq 1$, $xa^j y \in L(w, \phi)$.

In other words, repeated occurrences of a character $a \in \Sigma$ are without effect. Hence, in this case we need only consider simple strings. Thus if $\Box_i\phi \leftrightarrow \Box_i\Box_i\phi$ is a theorem for all formulas ϕ, then with out loss of generality we can assume that $L(w, \phi)$ are sets of simple strings. Hence, in order to restrict oneself to simple strings one must assume that the underlying logic contains the 4 axiom ($\Box_i\phi \to \Box_i\Box_i\phi$) and the secondary reflexivity axiom ($\Box_i(\Box_i\phi \to \phi)$).

In a similar way, downward closed sets corresonds to the truth axiom ($\Box_i\phi \to \phi$). Thus if the underlying logic is $S4$, then every set of the form $L(w, \phi)$ is downward closed and can be assumed to be simple.

3.2.1 Levels of Knowledge

3.2.2 Levels of Beliefs