# CISC 2210 – Introduction to Discrete Structures

## Midterm 1 Exam Solutions

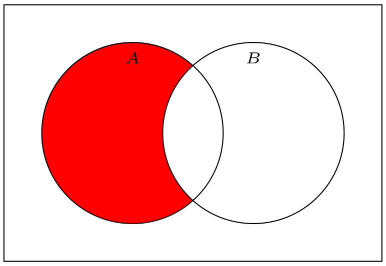
October 5, 2021

1. For two sets $A$ and $B$, find a simpler form for the sets $S$ and $T$.

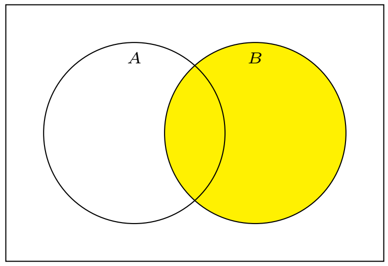
$$S = (A \setminus B) \cup B \qquad T = A \cup (B \setminus A)$$

**Answer:** $S = T = A \cup B$.

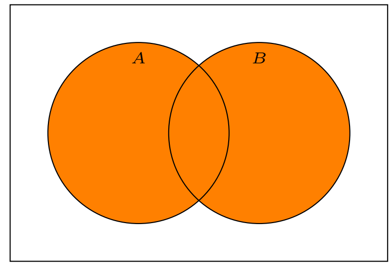
**Proof with Venn Diagrams:**



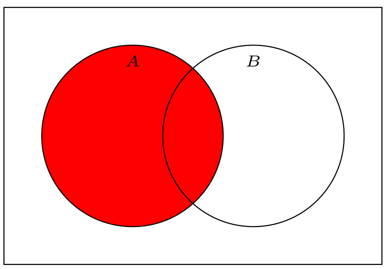

$A \setminus B$



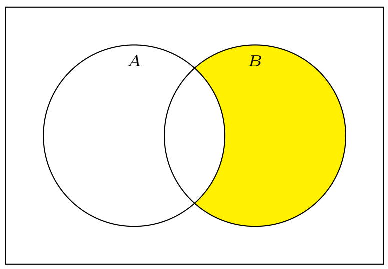

$B$



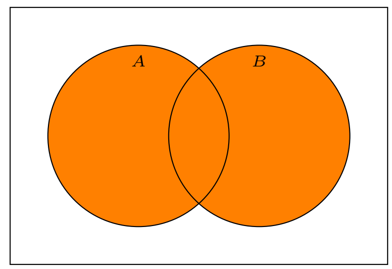

$S = (A \setminus B) \cup B = A \cup B$




$A$




$B \setminus A$




$T = A \cup (B \setminus A) = A \cup B$

**Proof II:** First observe that all the objects in both $S$ and $T$ are coming from $A$ or $B$ or both. As a result $S \subseteq A \cup B$ and $T \subseteq A \cup B$.

Next show that $x \in S$ and $x \in T$ for any $x \in A \cup B$ and therefore $A \cup B \subseteq S$ and $A \cup B \subseteq T$. There are three cases:

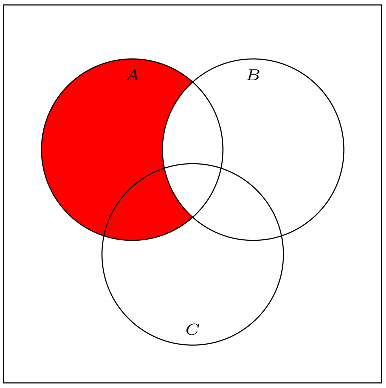
- $x \in A \setminus B$: Then $x \in S$ because $A \setminus B$ is a subset of $S$ and $x \in T$ because $A \setminus B$ is a subset of $A$ which is a subset of $T$.
- $x \in B \setminus A$: Then $x \in S$ because $B \setminus A$ is a subset of of $B$ which is a subset of $S$ and $x \in T$ because $B \setminus A$ is a subset of $T$.
- $x \in A \cap B$: Then $x \in S$ because $A \cap B$ is a subset of of $B$ which is a subset of $S$ and $x \in T$ because $A \cap B$ is a subset of of $A$ which is a subset of $T$.

Since for any two sets $X$ and $Y$, if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$, it follows that $S = T = A \cup B$.

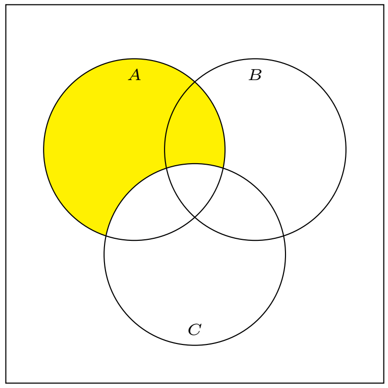
2. In the Venn Diagrams for the three sets $A$, $B$, and $C$, mark the area representing the set

$$(A \cap \overline{B}) \cup (A \cap \overline{C})$$

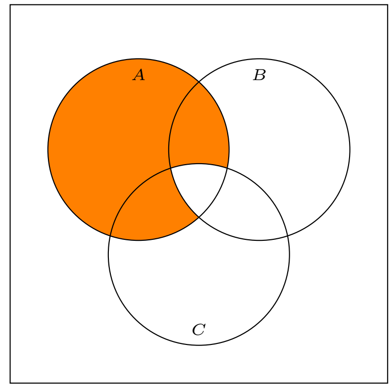
**Construction I:**




$A \cap \overline{B} = A \setminus B$




$A \cap \overline{C} = A \setminus C$

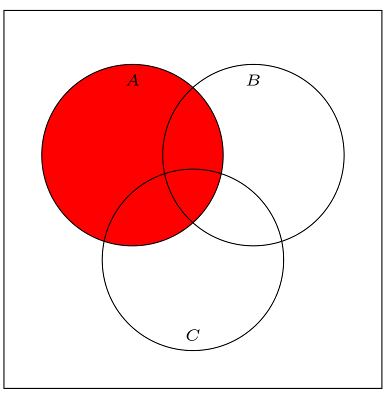



$(A \cap \overline{B}) \cup (A \cap \overline{C})$

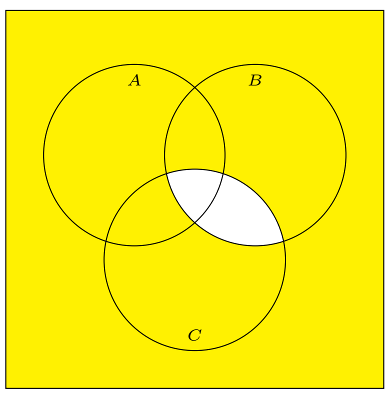
**Observation:** The distributive low and the De Morgan's law imply that

$$\begin{aligned}(A \cap \overline{B}) \cup (A \cap \overline{C}) &= A \cap (\overline{B} \cup \overline{C}) \\ &= A \cap \overline{B \cap C}\end{aligned}$$

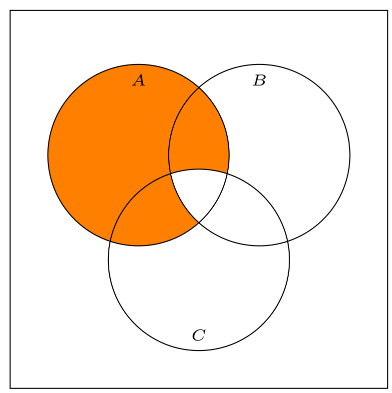
**Construction II:**




$A$




$\overline{B \cap C}$




$A \cap \overline{B \cap C}$

3. There are 100 students in the Computer Science program:
   - 30 students can program in C, 40 students can program in Java, and 50 students can program in Python.
   - Every student masters at least one language.
   - However, if a student knows more than one language, then this student knows all three of the languages.

   How many students can program in all three languages?

   **Answer: 10**.

   **Proof:** Let $x$ be the number of students who can program in all three languages. Since a student masters either exactly one language or all of them, it follows that $30 - x$ students can program only in C, $40 - x$ students can program only in Java, and $50 - x$ students can program only in Python. Therefore

$$\begin{aligned}
100 &= (30 - x) + (40 - x) + (50 - x) + x \\
&= 120 - 3x + x \\
&= 120 - 2x
\end{aligned}$$

   This implies that $x = 10$.

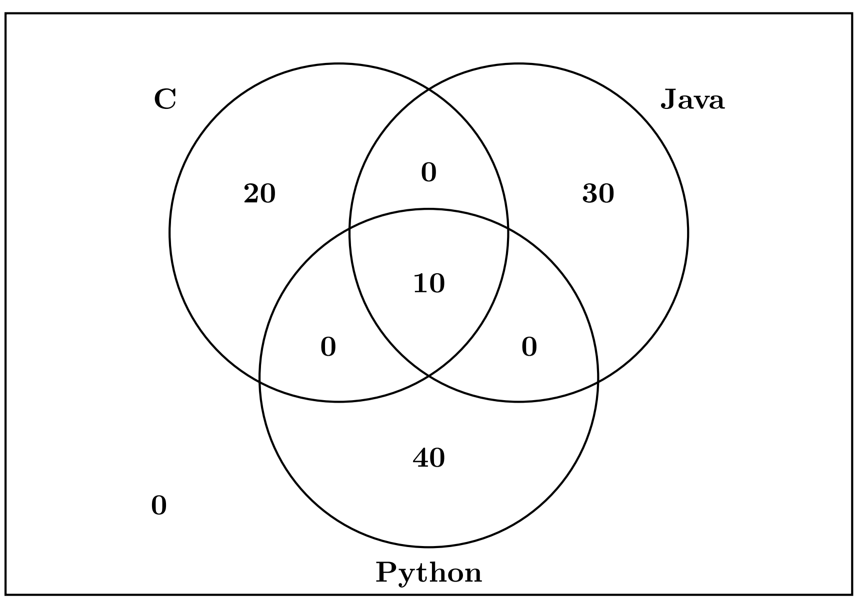
   **Proof with the Principle of Inclusion Exclusion:** Let $C$ be the set of students who can program in C, let $J$ be the set of students who can program in Java, and let $P$ be the set of students who can program in Python. The goal is to find the number of students in the set $C \cap J \cap P$. Let $x = |C \cap J \cap P|$.

   - Since every student masters at least one language, it follows that $|C \cup J \cup P| = 100$.
   - Since no student can master exactly two languages, it follows that $C \cap J = C \cap P = J \cap P = C \cap J \cap P$.

$$\begin{aligned}
100 &= |C \cup J \cup P| \\
&= |C| + |J| + |P| - |C \cap J| - |C \cap P| - |J \cap P| + |C \cap J \cap P| \\
&= 30 + 40 + 50 - x - x - x + x \\
&= 120 - 2x
\end{aligned}$$

   This implies that $x = 10$.

   See below the number of students in each of the 8 zones of the Venn Diagram for the sets $C$, $J$, and $P$.



4. In the following truth table, the columns $T_1, T_2, T_3, T_4, T_5$ represent 5 different formulas. Below you can find 4 formulas in an arbitrary order. Match each formula with one of the $T$ columns (one column does not represent any of the 4 formulas).

| $x$ | $y$ | $z$ | $w$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

**Formula 1:** $F_1 = (x \wedge y) \vee (z \wedge w)$

**Formula 2:** $F_2 = (x \vee w) \wedge (y \vee z)$

**Formula 3:** $F_3 = (\neg x \wedge y) \vee (\neg x \wedge z) \vee (x \wedge w)$

**Formula 4:** $F_4 = (x \wedge y) \oplus (z \vee w)$

**Answer:** The matching is

$$\begin{array}{ccc} F_1 & \longleftrightarrow & T_2 \\ F_2 & \longleftrightarrow & T_4 \\ F_3 & \longleftrightarrow & T_5 \\ F_4 & \longleftrightarrow & T_1 \end{array}$$

Observe that column $T_3$ has no match.

Three methods to solve this problem are described below. The third method relies on the fact that each formula matches one of the columns and two formulas cannot be matched with the same column. The second method still relies on the fact that each formula matches one of the columns, however it still works if more than one formula matches the same column. The first method works even if there are errors and some of the formulas have no match. In this case, this method identify the unmatched formula.

**Method I:** Construct the truth table of each formula and find the matches by comparing the results with the five $T$ columns.

- $F_1 = (x \wedge y) \vee (z \wedge w) \quad \longleftrightarrow \quad T_2$

| $x$ | $y$ | $z$ | $w$ | $x \wedge y$ | $z \wedge w$ | $(x \wedge y) \vee (z \wedge w)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

- $F_2 = (x \vee w) \wedge (y \vee z) \quad \longleftrightarrow \quad T_4$

| $x$ | $y$ | $z$ | $w$ | $x \vee w$ | $y \vee z$ | $(x \vee w) \wedge (y \vee z)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

- $F_3 = (\neg x \wedge y) \vee (\neg x \wedge z) \vee (x \wedge w) \quad \longleftrightarrow \quad T_5$

| $x$ | $y$ | $z$ | $w$ | $\neg x \wedge y$ | $\neg x \wedge z$ | $x \wedge w$ | $(\neg x \wedge y) \vee (\neg x \wedge z) \vee (x \wedge w)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

- $F_4 = (x \wedge y) \oplus (z \vee w) \quad \longleftrightarrow \quad T_1$

| $x$ | $y$ | $z$ | $w$ | $x \wedge y$ | $z \vee w$ | $(x \wedge y) \oplus (z \vee w)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

**Method II:** Independently, partially analyze each formula to identify the matching column. Try to examine as few as possible rows and/or sub-formulas. The following is one out of many ways to implement this method.

- $F_1 = (x \wedge y) \vee (z \wedge w) \quad \longleftrightarrow \quad T_2$
  **Proof:** $F_1 = T$ when $x = T$ and $y = T$. This means that the first four rows in the column must be $T$. Thus only $T_2$ and $T_4$ could be the match. $F_1 = F$ when both $x = F$ and $z = F$. Since $T_4 = T$ and $T_2 = F$ in row 11, it follows that the match must be $T_2$.

- $F_2 = (x \vee w) \wedge (y \vee z) \quad \longleftrightarrow \quad T_4$
  **Proof:** $F_2 = T$ when $x = T$ and $y = T$. This means that the first four rows in the column must be $T$. Thus only $T_2$ and $T_4$ could be the match. $F_2 = T$ when both $x = T$ and $z = T$. Since $T_2 = F$ and $T_4 = T$ in row 6, it follows that the match must be $T_4$.

- $F_3 = (\neg x \wedge y) \vee (\neg x \wedge z) \vee (x \wedge w) \quad \longleftrightarrow \quad T_5$
  **Proof:** $F_3 = T$ when $x = T$ and $w = T$. This means that the four rows $1, 3, 5, 7$ in the column must be $T$. This is the case only with $T_5$ which is the match.

- $F_4 = (x \wedge y) \oplus (z \vee w) \quad \longleftrightarrow \quad T_1$
  **Proof:** In the first four rows $(x \wedge y) = T$ and therefore $F_4 = T$ only if $(z \vee w) = F$. As a result, the first three rows in the column must be $F$ while the $4^{\text{th}}$ row must be $T$. This is the case only with $T_1$ which is the match.

**Method III:** Look in the truth table for a row that has either a singleton $T$ or a singleton $F$. Try to match this row with one of the formulas. If no match is found, eliminate this column and continue. If a match is found, eliminate this column and its matched formula and continue. The process ends when all formulas are matched.

- $F_4 = (x \wedge y) \oplus (z \vee w) \quad \longleftrightarrow \quad T_1$
  **Proof:** In row 3 only $T_1 = F$. Since in this row both $x = T$ and $y = T$, it follows that $F_1 = T$ and $F_2 = T$ and therefore both cannot be the match of $T_1$. $F_3$ cannot be the match of $T_1$ because $x = T$ and $w = F$ in row 4 which imply that all three sub-formulas of $F_3$ are False and therefore $F_3 = F$. It follows that $F_4$ must be the match of $T_1$. Note, that if $F_4$ is examined first, there would be no need to examine the other three formulas.
- $F_3 = (\neg x \wedge y) \vee (\neg x \wedge z) \vee (x \wedge w) \quad \longleftrightarrow \quad T_5$
  **Proof:** Without column $T_1$, in row 10 only $T_5 = T$. Since in this row both $x = F$ and $w = F$, it follows that $F_1 = F$ and $F_2 = F$ and therefore both cannot be the match of $T_1$. Since $F_4$ is already matched, the match of $T_5$ must be $F_3$.
- $F_2 = (x \vee w) \wedge (y \vee z) \quad \longleftrightarrow \quad T_4$
  **Proof:** Without $T_1$ and $T_5$, in row 6 only $T_4 = T$. Since in this row both $x = T$ and $z = T$, it follows that $F_2 = T$. As a result, the match of $T_4$ must be $F_2$.
- $F_1 = (x \wedge y) \vee (z \wedge w) \quad \longleftrightarrow \quad T_2$
  **Proof:** At this stage, the only unmatched formula is $F_1$ and the only unmatched columns are $T_2$ and $T_3$. In the first row $F_1 = T$ because $x = T$ and $y = T$ and therefore the match of $F_1$ must be $T_2$.

**Remark:** The above is one out of many ways to implement this method. If at any stage, there is a column that is not matched with any of the formulas, this column can be eliminated and the process proceeds without it.

5. For $n \geq 1$, the goal is to count the number of truth assignments that satisfy the following formula on the $n+1$ variables $x, x_1, x_2, \ldots, x_n$:

$$x \oplus (x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_n)$$

Recall that in total there are $2^{n+1}$ possible assignments.

(a) The case $n = 1$. How many truth assignments satisfy the formula: $x \oplus x_1$?
**Answer:** 2 assignments.
**Proof:** If $x$ is True then $x_1$ must be False and if $x$ is False then $x_1$ must be True. It follows that the truth assignments are the following two assignments:

$$(x, x_1) \longleftarrow \quad (TF) \quad (FT)$$

(b) The case $n = 2$. How many truth assignments satisfy the formula: $x \oplus (x_1 \wedge x_2)$?
**Answer:** 4 assignments.
**Proof:** If $x$ is True then $x_1 \wedge x_2$ must be False and this happens when

$$(x_1, x_2) \longleftarrow \quad (TF) \quad (FT) \quad (FF)$$

If $x$ is False then $x_1 \wedge x_2$ must be True and this happens only when

$$(x_1, x_2) \longleftarrow \quad (TT)$$

It follows that the truth assignments are the following four assignments:

$$(x, x_1, x_2) \longleftarrow \quad (TTF) \quad (TFT) \quad (TFF) \quad (FTT)$$

(c) The case $n = 3$. How many truth assignments satisfy the formula: $x \oplus (x_1 \wedge x_2 \wedge x_3)$?
**Answer:** 8 assignments.
**Proof:** If $x$ is True then $x_1 \wedge x_2 \wedge x_3$ must be False and this happens when

$$(x_1, x_2, x_3) \longleftarrow \quad (TTF) \quad (TFT) \quad (TFF) \quad (FTT) \quad (FTF) \quad (FFT) \quad (FFF)$$

If $x$ is False then $x_1 \wedge x_2 \wedge x_3$ must be True and this happens only when

$$(x_1, x_2, x_3) \longleftarrow \quad (TTT)$$

It follows that the truth assignments are the following eight assignments:

$$(x, x_1, x_2, x_3) \longleftarrow \quad (TTTF) \quad (TTFT) \quad (TTFF) \quad (TFTT) \quad (TFTF) \quad (TFFT) \quad (TFFF) \quad (FTTT)$$

(d) The general case. How many truth assignments satisfy the formula: $x \oplus (x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_n)$?
**Answer:** $2^n$ assignments.
**Proof:** If $x$ is True then $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ must be False and this happens in the $2^n - 1$ truth assignments to $x_1, x_2, \ldots, x_n$ in which at least one of them is False. If $x$ is False then $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ must be True and this happens only when all of the $n$ variables $x_1, x_2, \ldots, x_n$ are True. It follows that there are $2^n = (2^n - 1) + 1$ truth assignments.

**Remark:** The answer is $2^n$ for any formula $x \oplus P$ in which the sub-formula $P$ contains $n$ additional variables. Assume $P$ has $k$ truth assignments and $n - k$ non-truth assignments. Then in any truth assignment to $x \oplus P$, when $P$ is True $x$ must be False and when $P$ is False $x$ must be True. Therefore, $x \oplus P$ has in total $2^n = k + (2^n - k)$ truth assignments. In the question $P = x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for which $k = 1$ and $n - k = 2^n - 1$.

6. In front of you there are three jars:

   - One jar contains only Red marbles, one jar contains only Blue marbles, and one jar contains only Green marbles.
   - One jar is labeled Red/Blue, one is labeled Blue/Green, and one is labeled Green/Red.
   - A jar that is labeled with two colors must contain one of them, but it is unknown which one.

   You are allowed to select one marble from any jar of your choice and observe its color. Then you must correct the labels to accurately indicate the color of the marbles in each jar.

   Describe how to accomplish this task and explain why your method is always correct.

   **Answer:** Select a marble from the Red/Blue jar.

   - If the selected marble is Red then you learn that this jar contains Red marbles. Moreover, this implies that the Red/Green labeled jar must contain Green marbles because the Red marbles are in the selected jar and that the Blue/Green labeled jar must contain Blue marbles because this is the only jar left for the Blue marbles.
   - If the selected marble is Blue then you learn that this jar contains Blue marbles. Moreover, this implies that the Blue/Green labeled jar must contain Green marbles because the Blue marbles are in the selected jar and that the Red/Green labeled jar must contain Red marbles because this is the only jar left for the Red marbles.

   **Remark:** You may select any of the three jars. The strategy for deciding the correct labels is similar.