CISC 2210 – Introduction to Discrete Structures

Midterm 1 Exam Solutions

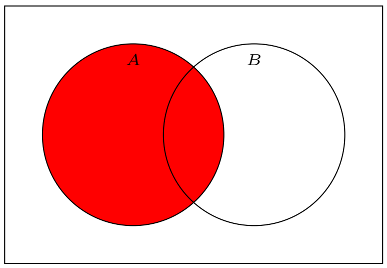
October 5, 2021

1. For two sets A and B, find a simpler form for the sets S and T.

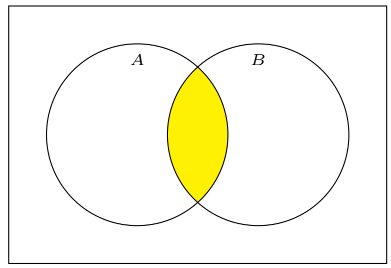
$$S = A \cup (A \cap B) \qquad T = A \cap (A \cup B)$$

Answer: $S = T = A$.

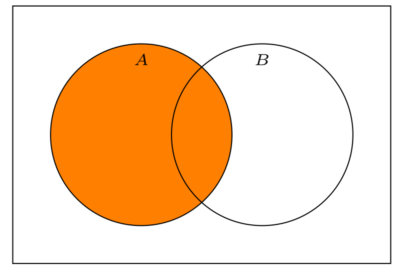
Proof with Venn Diagrams:



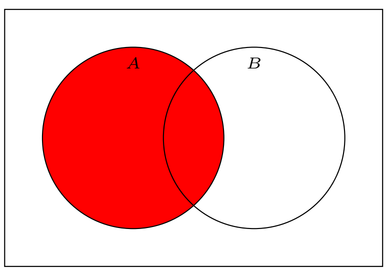
A



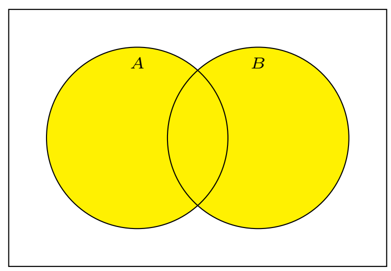
$A \cap B$



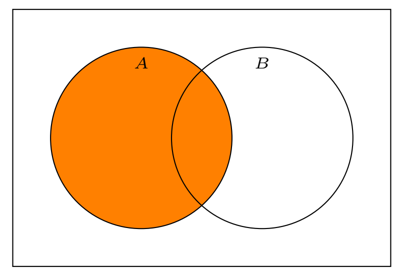
$S = A \cup (A \cap B) = A$



A



$A \cup B$



$T = A \cap (A \cup B) = A$

Proof II: Observe that for two sets X and Y, if $Y \subseteq X$ then (i) $Y \cup X = X$ and (ii) $Y \cap X = Y$.

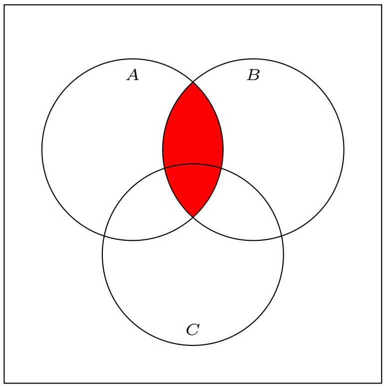
- By definition, $A \cap B \subseteq A$ and therefore (i) implies that $S = A \cup (A \cap B) = A$.
- By definition, $A \subseteq A \cup B$ and therefore (ii) implies that $T = A \cap (A \cup B) = A$.

Proof III: To prove that $S = T = A$, the following shows that if $x \in A$ then $x \in S$ and $x \in T$ and that if $x \in S$ or $x \in T$ then $x \in A$.

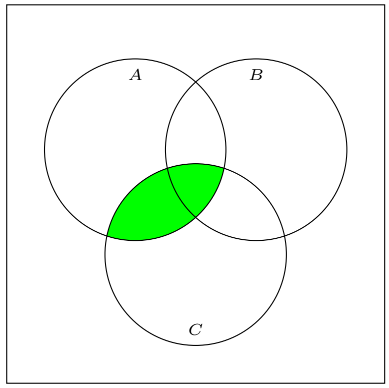
- Assume $x \in A$, then $x \in S$ because S is a union of A with another set.
- Assume $x \in A$, then $x \in T$ because $x \in A \cup B$ and T is an intersection of two sets that contain x.
- Assume $x \in S$, then either $x \in A$ or $x \in A \cap B$. In both cases $x \in A$.
- Assume $x \in T$, then $x \in A$ because T is an intersection of A with another set.

2. In the Venn Diagrams for the three sets A, B, and C, mark the area representing the set

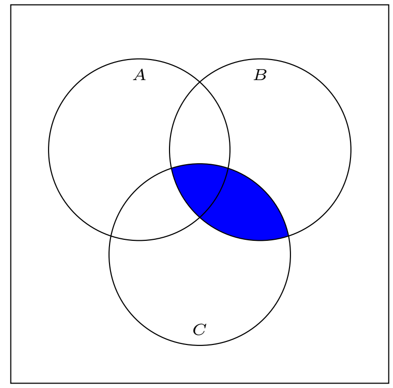
$$((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$$



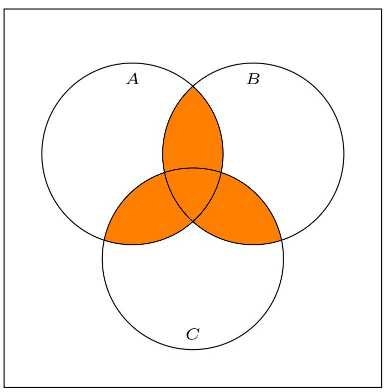
$A \cap B$



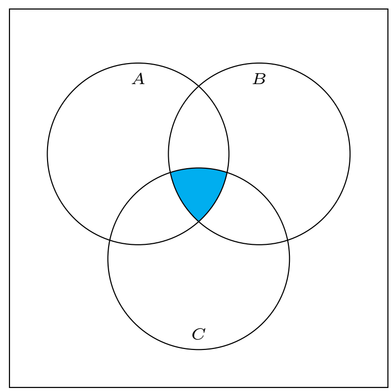
$A \cap C$



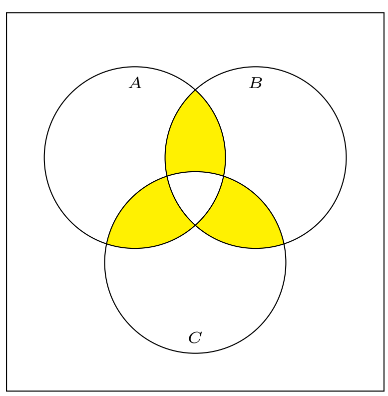
$B \cap C$



$(A \cap B) \cup (A \cap C) \cup (B \cap C)$



$A \cap B \cap C$



$((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$

3. After grading the Midterm exam, the professor created the following three lists of students:
   - One list contained the names of all the 30 students whose grades were between 0 and 60.
   - One list contained the names of all the 30 students whose grades were between 40 and 100.
   - One list contained the names of all the 20 students whose grades were between 30 and 70.
   - 15 students appeared in exactly two lists.

   Answer the following two questions.

   (a) How many students appeared in all three lists?

   (b) How many students took the exam?

   Answer: **5** students out of the **55** students who took the exam appeared in all three lists.

   Proof: Let A be the set containing the names of the 30 students whose grades were between 0 and 60 ($|A| = 30$), let B be the set containing the names of the 30 students whose grades were between 40 and 100 ($|B| = 30$), and let C be the set containing the names of the 20 students whose grades were between 30 and 70 ($|C| = 20$).

   Observation I: $C \subseteq A \cup B$.

   Observation II: $(A \cap B) \subseteq C$ and therefore $A \cap B = A \cap B \cap C$.

   Observation II implies that every students who appeared in more than one list must appear in C while Observation I implies that C includes only students who appeared in two or three lists.

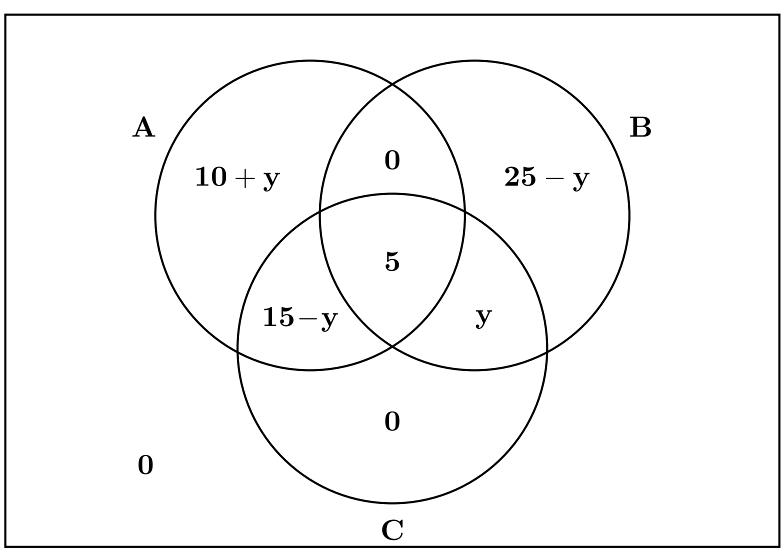
   Since there were 15 students who appeared in exactly two lists and since the size of C is 20, it follows that $\mathbf{5} = 20 - 15$ students appeared in all three lists.

   By definition, A and B together contain all the students who took the exam. Therefore the number of students who took the exam is the size of the set $A \cup B$. The Principle of Inclusion Exclusion for the sets A and B, Observation II, and the previously computed size of the set $A \cap B$ imply that

   $$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 30 - 5 = 55$$

   Discussion: Denote by y the number of students who belong to B and C but not to A. Since there are 15 students who appeared in exactly two lists and since by Observation II there were no students who belong to A and B but not in C, it follows that $15 - y$ is the number of students who belong to A and C but not in B.

   See below the number of students in each of the 8 zones of the Venn Diagram for the sets A, B, and C. Note that there could be 16 different possibilities depending on the value of y from 0 to 15. However, in all of them 5 students out of the 55 ((10+y)+(25-y)+0+0+y+(15-y)+5) students who took the exam appeared in all three lists.



4. In the following truth table, the columns T_1, T_2, T_3, T_4, T_5 represent 5 different formulas. Below you can find 4 formulas in an arbitrary order. Match each formula with one of the T columns (one column does not represent any of the 4 formulas).

x	y	z	w	T_1	T_2	T_3	T_4	T_5
T	T	T	T	T	F	T	T	F
T	T	T	F	T	T	T	T	T
T	T	F	T	F	T	T	T	T
T	T	F	F	F	F	F	F	T
T	F	T	T	F	T	T	T	F
T	F	T	F	F	F	F	T	T
T	F	F	T	F	F	T	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	F
F	T	T	F	F	F	T	T	T
F	T	F	T	T	F	F	T	T
F	T	F	F	F	T	F	F	T
F	F	T	T	T	F	F	F	T
F	F	T	F	F	T	F	F	F
F	F	F	T	T	T	F	F	F
F	F	F	F	F	F	F	F	F

Formula 1: $F_1 = (x \vee y) \wedge (z \vee w)$

Formula 2: $F_2 = (x \wedge w) \vee (y \wedge z)$

Formula 3: $F_3 = (\neg x \vee y) \wedge (\neg x \vee z) \wedge (x \vee w)$

Formula 4: $F_4 = (x \vee y) \oplus (z \wedge w)$

Answer: The matching is

$$\begin{aligned} F_1 &\longleftrightarrow T_4 \\ F_2 &\longleftrightarrow T_3 \\ F_3 &\longleftrightarrow T_1 \\ F_4 &\longleftrightarrow T_5 \end{aligned}$$

Observe that column T_2 has no match.

Three methods to solve this problem are described below. The third method relies on the fact that each formula matches one of the columns and two formulas cannot be matched with the same column. The second method still relies on the fact that each formula matches one of the columns, however it still works if more than one formula matches the same column. The first method works even if there are errors and some of the formulas have no match. In this case, this method identify the unmatched formulas.

Method I: Construct the truth table of each formula and find the matches by comparing the results with the five T columns.

- $F_1 = (x \vee y) \wedge (z \vee w) \quad \longleftrightarrow \quad T_4$

x	y	z	w	$x \vee y$	$z \vee w$	$(x \vee y) \wedge (z \vee w)$
T	T	T	T	T	T	T
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	T	F	F	T	F	F
F	F	T	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	T	F
F	F	F	F	F	F	F

- $F_2 = (x \wedge w) \vee (y \wedge z) \quad \longleftrightarrow \quad T_3$

x	y	z	w	$x \wedge w$	$y \wedge z$	$(x \wedge w) \vee (y \wedge z)$
T	T	T	T	T	T	T
T	T	T	F	F	T	T
T	T	F	T	T	F	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	T	F	F	F	F
T	F	F	T	T	F	T
T	F	F	F	F	F	F
F	T	T	T	F	T	T
F	T	T	F	F	T	T
F	T	F	T	F	F	F
F	T	F	F	F	F	F
F	F	T	T	F	F	F
F	F	T	F	F	F	F
F	F	F	T	F	F	F
F	F	F	F	F	F	F

- $F_3 = (\neg x \vee y) \wedge (\neg x \vee z) \wedge (x \vee w) \quad \longleftrightarrow \quad T_1$

x	y	z	w	$\neg x \vee y$	$\neg x \vee z$	$x \vee w$	$(\neg x \vee y) \wedge (\neg x \vee z) \wedge (x \vee w)$
T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	T
T	T	F	T	T	F	T	F
T	T	F	F	T	F	T	F
T	F	T	T	F	T	T	F
T	F	T	F	F	T	T	F
T	F	F	T	F	F	T	F
T	F	F	F	F	F	T	F
F	T	T	T	T	T	T	T
F	T	T	F	T	T	F	F
F	T	F	T	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	T	T	T	T	T
F	F	T	F	T	T	F	F
F	F	F	T	T	T	T	T
F	F	F	F	T	T	F	F

- $F_4 = (x \vee y) \oplus (z \wedge w) \quad \longleftrightarrow \quad T_5$

x	y	z	w	$x \vee y$	$z \wedge w$	$(x \vee y) \oplus (z \wedge w)$
T	T	T	T	T	T	F
T	T	T	F	T	F	T
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	T	T	F
T	F	T	F	T	F	T
T	F	F	T	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	F
F	T	T	F	T	F	T
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	F	T	T
F	F	T	F	F	F	F
F	F	F	T	F	F	F
F	F	F	F	F	F	F

Method II: Independently, partially analyze each formula to identify the matching column. Try to examine as few as possible rows and/or sub-formulas. The following is one out of many ways to implement this method.

- $F_1 = (x \vee y) \wedge (z \vee w) \quad \longleftrightarrow \quad T_4$
  Proof: $F_1 = F$ when $x = F$ and $y = F$. This means that the last four rows in the column must be F. Thus only T_3 and T_4 could be the match. $F_1 = T$ when both $x = T$ and $z = T$. Since $T_4 = T$ and $T_2 = F$ in row 6, it follows that the match must be T_4.

- $F_2 = (x \wedge w) \vee (y \wedge z) \quad \longleftrightarrow \quad T_3$
  Proof: $F_2 = F$ when $x = F$ and $y = F$. This means that the last four rows in the column must be F. Thus only T_3 and T_4 could be the match. $F_2 = F$ when both $x = F$ and $z = F$. Since $T_3 = F$ and $T_4 = T$ in row 11, it follows that the match must be T_3.

- $F_3 = (\neg x \vee y) \wedge (\neg x \vee z) \wedge (x \vee w) \quad \longleftrightarrow \quad T_1$
  Proof: $F_3 = F$ when $x = F$ and $w = F$. This means that the four rows $10, 12, 14, 16$ in the column must be F. This is the case only with T_1 which is the match.

- $F_4 = (x \vee y) \oplus (z \wedge w) \quad \longleftrightarrow \quad T_5$
  Proof: In the last four rows $(x \vee y) = F$ and therefore $F_4 = T$ only if $(z \wedge w) = T$. As a result, the last three rows in the column must be F while the 13^{th} row must be T. This is the case only with T_5 which is the match.

Method III: Look in the truth table for a row that has either a singleton T or a singleton F. Try to match this row with one of the formulas. If no match is found, eliminate this column and continue. If a match is found, eliminate this column and its matched formula and continue. The process ends when all formulas are matched.

- $F_4 = (x \vee y) \oplus (z \wedge w) \quad \longleftrightarrow \quad T_5$
  Proof: In row 4 only $T_5 = T$. Since in this row both $z = F$ and $w = F$, it follows that $F_1 = F$ and $F_2 = F$ and therefore both cannot be the match of T_5. F_3 cannot be the match because $x = T$ and $z = F$ and therefore $F_3 = F$. It follows that F_4 must be the match of T_5. Note, that if F_4 is examined first, there would be no need to examine the other three formulas.
- $F_3 = (\neg x \vee y) \wedge (\neg x \vee z) \wedge (x \vee w) \quad \longleftrightarrow \quad T_1$
  Proof: Without column T_5, in row 13 only $T_1 = T$. Since in this row both $x = F$ and $y = F$, it follows that $F_1 = F$ and $F_2 = F$ and therefore both cannot be the match of T_1. Since F_4 is already matched, the match of T_1 must be F_3.
- $F_1 = (x \vee y) \wedge (z \vee w) \quad \longleftrightarrow \quad T_4$
  Proof: Without T_1 and T_5, in row 11 only $T_4 = T$. Since in this row both $y = T$ and $w = T$, it follows that $F_1 = T$. As a result, the match of T_4 must be F_1.
- $F_2 = (x \wedge w) \vee (y \wedge z) \quad \longleftrightarrow \quad T_3$
  Proof: At this stage, the only unmatched formula is F_2 and the only unmatched columns are T_2 and T_3. In the first row $F_1 = T$ because $x = T$ and $y = T$ and therefore the match of F_1 must be T_3.

Remark: The above is one out of many ways to implement this method. If at any stage, there is a column that is not matched with any of the formulas, this column can be eliminated and the process proceeds without it.

5. For $n \geq 1$, the goal is to count the number of truth assignments that satisfy the following formula on the $n+1$ variables $y, y_1, y_2, \ldots, y_n$:

$$y \equiv (y_1 \vee y_2 \vee y_3 \vee \cdots \vee y_n)$$

Recall that in total there are 2^{n+1} possible assignments.

(a) The case $n = 1$. How many truth assignments satisfy the formula: $y \equiv y_1$?
Answer: 2 assignments.
Proof: If y is True then y_1 must be True and if y is False then y_1 must be False. It follows that the truth assignments are the following two assignments:

$$(y, y_1) \quad \longleftarrow \quad (TT) \quad (FF)$$

(b) The case $n = 2$. How many truth assignments satisfy the formula: $y \equiv (y_1 \vee y_2)$?
Answer: 4 assignments.
Proof: If y is True then $y_1 \vee y_2$ must be True and this happens when

$$(y_1, y_2) \quad \longleftarrow \quad (TT) \quad (TF) \quad (FT)$$

If y is False then $y_1 \vee y_2$ must be False and this happens only when

$$(y_1, y_2) \quad \longleftarrow \quad (FF)$$

It follows that the truth assignments are the following four assignments:

$$(y, y_1, y_2) \quad \longleftarrow \quad (TTT) \quad (TTF) \quad (TFT) \quad (FFF)$$

(c) The case $n = 3$. How many truth assignments satisfy the formula: $y \equiv (y_1 \vee y_2 \vee y_3)$?
Answer: 8 assignments.
Proof: If y is True then $y_1 \vee y_2 \vee y_3$ must be True and this happens when

$$(y_1, y_2, y_3) \quad \longleftarrow \quad (TTT) \quad (TTF) \quad (TFT) \quad (TFF) \quad (FTT) \quad (FTF) \quad (FFT)$$

If y is False then $y_1 \vee y_2 \vee y_3$ must be False and this happens only when

$$(y_1, y_2, y_3) \quad \longleftarrow \quad (FFF)$$

It follows that the truth assignments are the following eight assignments:

$$(y, y_1, y_2, y_3) \quad \longleftarrow \quad (TTTT) \quad (TTTF) \quad (TTFT) \quad (TTFF) \quad (TFTT) \quad (TFTF) \quad (TFFT) \quad (FFFF)$$

(d) The general case. How many truth assignments satisfy the formula: $y \equiv (y_1 \vee y_2 \vee y_3 \vee \cdots \vee y_n)$?
Answer: 2^n assignments.
Proof: If y is True then $y_1 \vee y_2 \vee \cdots \vee y_n$ must be True and this happens in the $2^n - 1$ truth assignments to $y_1, y_2, \ldots, y_n$ in which at least one of them is True. If y is False then $y_1 \vee y_2 \vee \cdots \vee y_n$ must be False and this happens only when all of the n variables $y_1, y_2, \ldots, y_n$ are False. It follows that there are $2^n = (2^n - 1) + 1$ truth assignments.

Remark: The answer is 2^n for any formula $y \equiv P$ in which the sub-formula P contains n additional variables. Assume P has k truth assignments and $n - k$ non-truth assignments. Then in any truth assignment to $y \equiv P$, when P is True y must be True and when P is False y must be False. Therefore, $y \equiv P$ has in total $2^n = k + (2^n - k)$ truth assignments. In the question, $P = y_1 \vee y_2 \vee \cdots \vee y_n$ for which $k = 2^n - 1$ and $n - k = 1$.

6. In front of you, on a table, you see 3 cards: One card is Blue while the other 2 cards are Red.

   Your goal is to identify the colors on the sides of the cards that you cannot see.

   You know that one card is Red on both sides, one card is Blue on both sides, and one card is Red on one side and Blue on the other side.

   You are allowed to select only one of the cards and observe the color of its other side.

   Which card will you select to learn the colors of the other side of all three cards.

   Describe your strategy and explain why it always works correctly.

   Answer: Observe first that only the Blue card can be the card with Blue on both sides because at least one side of the other two cards is Red. Therefore, you should not select the Blue card.

   After selecting one of the Red cards you know if the selected card is Red on both sides or Red on one side and Blue on the other side. This leaves the third option for the unselected Red card.