CISC 2210 – Introduction to Discrete Structures

Midterm 1 Exam Solutions

October 6, 2022

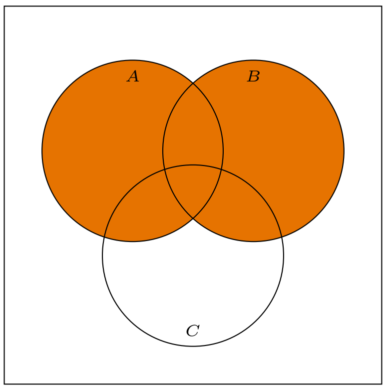
1. **Theorem:** $(A \cup B) \setminus C \equiv (A \setminus C) \cup (B \setminus C)$

   Proof: Let $L = (A \cup B) \setminus C$ and $R = (A \setminus C) \cup (B \setminus C)$. Consider the following two cases:

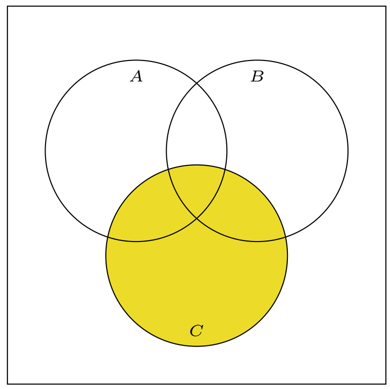
   - $x \in L$: By definition, $x \in A \cup B$ but $x \notin C$. Consequently, either $x \in A \setminus C$ or $x \in B \setminus C$. It follows that $x \in R$.
   - $y \in R$: By definition, either $y \in A \setminus C$ or $y \in B \setminus C$. In both cases $y \in A \cup B$ but $y \notin C$. It follows that $y \in R$.

   The above two cases show that every object of L belongs to R and every object of R belongs to L. As a result $L \equiv R$.

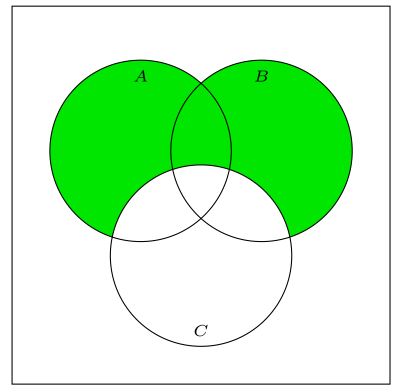
   Proof with Venn Diagrams:



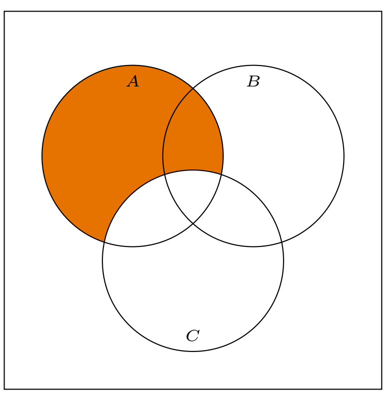
$A \cup B$



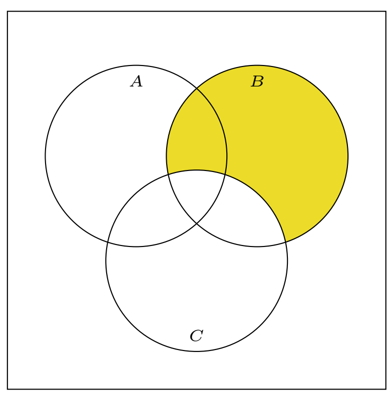
C



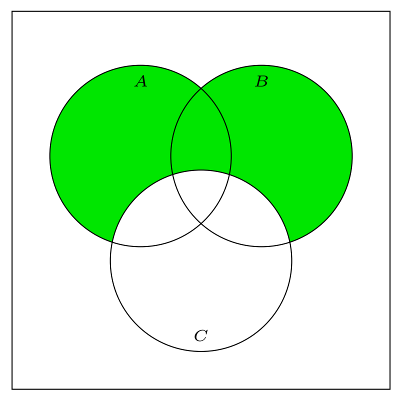
$(A \cup B) \setminus C$



$A \setminus C$



$B \setminus C$

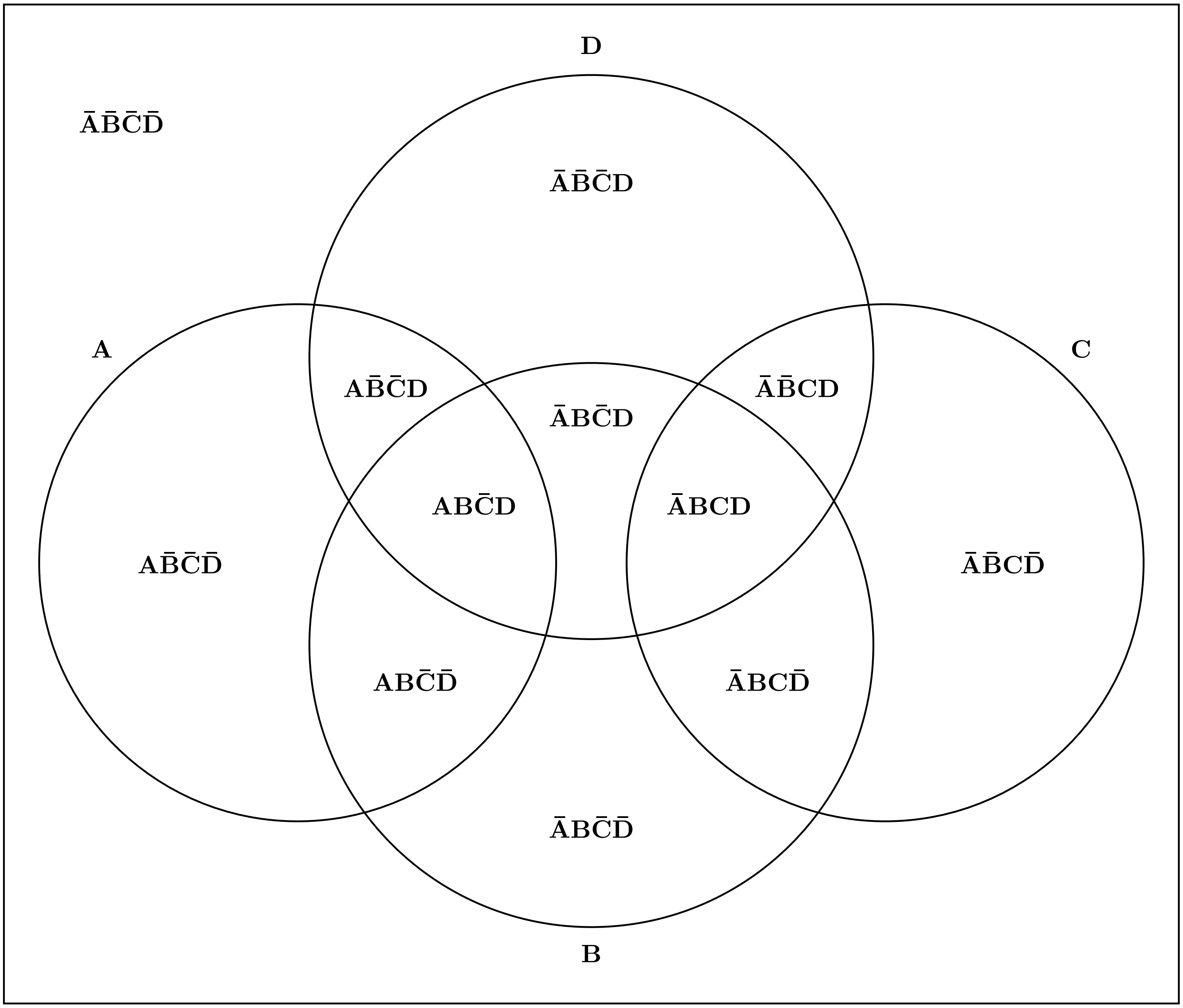


$(A \setminus C) \cup (B \setminus C)$

2. For any set S, denote by $\bar{S}$ the complement $\neg S$ of the set S. For any two sets S and T, denote by ST the intersection $S \cap T$ between the two sets.

   Example: The intersection $\neg A \cap B \cap \neg C \cap D$ is denoted by $\bar{A}B\bar{C}D$.

   Below is a Venn Diagram for some sets A, B, C, and D in which the twelve existing zones are marked.



   The zones that do not appear in the diagram are all the zones that require showing an intersection between the sets A and C. The four zone are:

$$ABCD \quad ABC\bar{D} \quad A\bar{B}CD \quad A\bar{B}C\bar{D}$$

   Note that the markings of the above four zones contain both A and C (equivalently, neither $\bar{A}$ nor $\bar{C}$). Indeed there are 4 options for B and D: BD, $B\bar{D}$, $\bar{B}D$, and $\bar{B}\bar{D}$.

3. Last week **36** students attended the discrete structures class: **14** students wore black shirts, **13** students wore black pants, and **12** students wore black shoes.

   5 students wore a black shirt and black pants, **3** students wore a black shirt and black shoes, **4** students wore black pants and black shoes, while only **1** student wore a black shirt, black pants, and black shoes.

   (a) How many students did not wear a black shirt, black pants, or black shoes?

   (b) How many students wore a black shirt but did not wear black pants or black shoes?

   (c) How many students wore black pants but did not wear a black shirt or black shoes?

   (d) How many students wore black shoes but did not wear a black shirt or black pants?

Processing the input: Let T be the set of all students who wore a black shirt, let P be the set of students who wore black pants, and let S be the set of students who wore black shoes. The provided numbers imply the following:

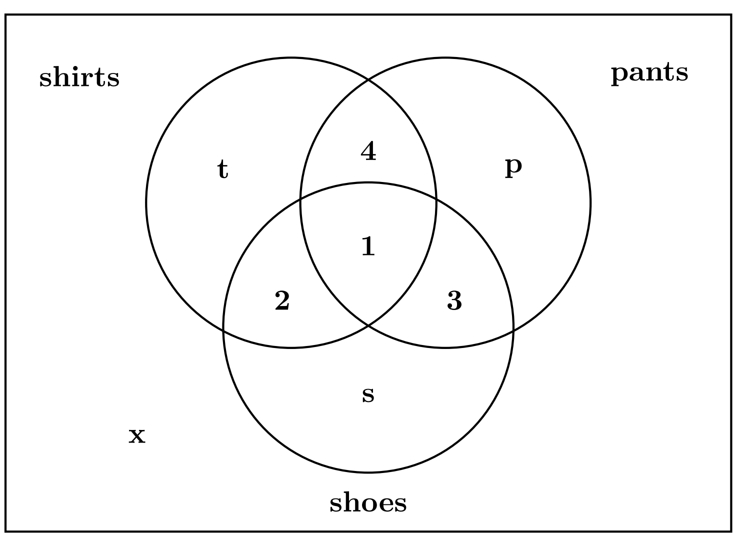
- $|T| = 14$, $|P| = 13$, and $|S| = 12$.
- $|T \cap P| = 5$, $|T \cap S| = 3$, and $|P \cap S| = 4$.
- $|T \cap P \cap S| = 1$.

The provided numbers also imply the following:

- 5 students wore a black shirt and black pants but only one of them wore also black shoes. Thus, 4 students wore a black shirt and black pants but did not ware black shoes: $|(T \cap P) \setminus S| = 4$.
- 3 students wore a black shirt and black shoes but only one of them wore also black pants. Thus 2 students wore a black shirt and black shoes but didn't ware black pants: $|(T \cap S) \setminus P| = 2$.
- 4 students wore black pants and black shoes but only one of them wore also a black shirt. Thus 3 students wore black pants and black shoes but didn't ware a black shirt: $|(P \cap S) \setminus T| = 3$.

Denote by $x = |\overline{T \cup P \cup S}|$ the number of students who did not wear a black shirt, black pants, or black shoes (the answer to part (a)), by t the number of students who wore a black shirt but did not wear black pants or black shoes (the answer to part (b)), by p the number of students who wore black pants but did not wear a black shirt or black shoes (the answer to part (c)), and by s the number of students who wore black shoes but did not wear a black shirt or black pants (the answer to part (d)).

The Venn Diagram below shows the number of students in each of the possible 8 zones using the variables x, t, p, and s.



Solution to part (a): By the principle of inclusion exclusion

$$
\begin{aligned}
|T \cup P \cup S| &= |T| + |P| + |S| - |T \cap P| - |T \cap S| - |P \cap S| + |T \cap P \cap S| \\
&= 14 + 13 + 12 - 5 - 3 - 4 + 1 \\
&= 28
\end{aligned}
$$

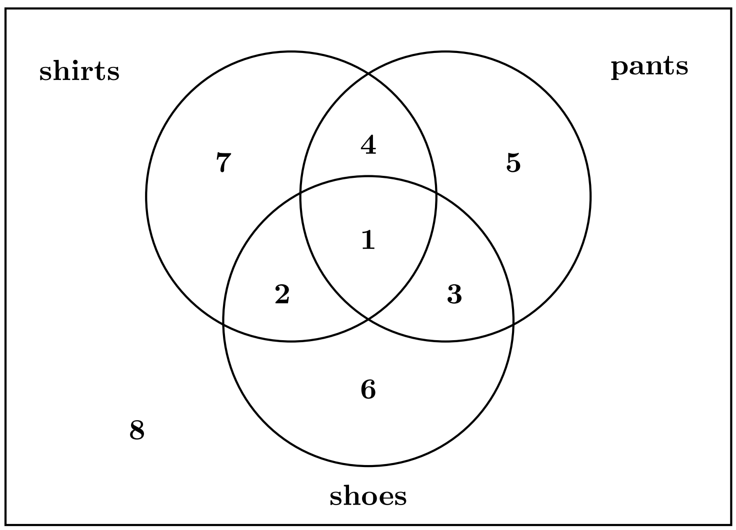
Since 36 students attended the discrete structures class, it follows that $x = 36 - |T \cup P \cup S| = 36 - 28 = 8$ students did not wear a black shirt, black pants, or black shoes.

Solution to part (b): Since $|T| = 14$, it follows that $t = 14 - 4 - 2 - 1 = 14 - 7 = 7$ students wore a black shirt but did not ware black pants or black shoes.

Solution to part (c): Since $|P| = 13$, it follows that $p = 13 - 4 - 3 - 1 = 13 - 8 = 5$ students wore black pants but did not ware a black shirt or black shoes.

Solution to part (d): Since $|S| = 12$, it follows that $s = 12 - 2 - 3 - 1 = 12 - 6 = 6$ students wore black shoes but did not ware a black shirt or black pants.

The Venn Diagram below shows the final numbers in all 8 zones.



4. Define a boolean formula $\mathcal{P}$ on the variables x, y, and z for which the following table is its truth table.

   Optimization goal: Find as short as possible formula.

x	y	z	$\mathcal{P}$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

   Answer: The truth table shows that $\mathcal{P}$ is TRUE either when $x = T$ or when both $y = T$ and $z = T$. That is, $P \equiv x \vee (y \wedge z)$. See the truth table below.

x	y	z	$y \wedge z$	$x \vee (y \wedge z)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

   Remark: One of the distributive laws for the functions $\mathcal{AND}$ ($\wedge$) and $\mathcal{OR}$ ($\vee$) is

$$x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$$

   Therefore $(x \vee y) \wedge (x \vee z)$ is also a formula for which the input table is its truth table. However, the formula $x \vee (y \wedge z)$ is shorter.

5. Let $X = \{1, 3, 5, 7, 9\}$, $Y = \{2, 4, 6, 8, 10\}$, and $Z = \{10, 11, 12, 13, 14\}$.

(a) The expression $\forall_{x\in X}\forall_{y\in Y}\exists_{z\in Z}\,\{x + y < z\}$ is **FALSE**.

Proof: Observe that if the expression is TRUE for the largest possible values of x and y then it must be TRUE for all $x \in X$ and for all $y \in Y$. However, $x + y = 19$ for $x = 9$ and $y = 10$ which is larger than any possible value of z. This counterexample shows that the expression is FALSE.

Remark: In fact any selection of $x \in X$ and $y \in Y$ such that $x + y \geq 14$ implies a counterexample because the largest possible value for z is $z = 14$. There are 6 such (x, y) pairs (out of 25 possible pairs) that imply counterexamples: $(5, 10)$, $(7, 8)$, $(7, 10)$, $(9, 6)$, $(9, 8)$, and $(9, 10)$.

(b) The expression $\exists_{x\in X}\forall_{y\in Y}\exists_{z\in Z}\,\{x + y > z\}$ is **TRUE**.

Proof: Observe that since the sum $x + y$ is required to be larger then z, the best strategy to prove that the expression is TRUE is to select the largest possible x and the smallest possible z. That is, select $x = 9$ and $z = 10$. Since $y \geq 2$ for all $y \in Y$, it follows that $x + y = 9 + y \geq 9 + 2 = 11 > z$.

Remark: Note that because the expression must be true for $y = 2$, it follows that $x = 9$ and $z = 10$ are the only possible selections of $x \in X$ and $z \in Z$ that imply that the expression is TRUE. This is because $x + 2 \leq 9 < 10 \leq z$ for $x \leq 7$ and for all $z \in Z$ and $x + 2 \leq 9 + 2 = 11 \leq z$ for $z \geq 11$ and for all $x \in X$.

(c) The expression $\exists_{z\in Z}\forall_{x\in X}\exists_{y\in Y}\,\{x + y = z\}$ is **TRUE**.

Proof: Select $z = 11$. It follows that $1 + 10 = 3 + 8 = 5 + 6 = 7 + 4 = 9 + 2 = 11$ and therefore for any selection of $x \in X$ there exists $y \in Y$ such that $x + y = 11$.

Remark: Note that any other selection of z cannot prove that the expression is TRUE. If z is even, the expression is FALSE since all the integers in X are odd, all the integers in Y are even, and the sum of an odd number and an even number must be odd. The other odd value for z is $z = 13$. In this case, $12 \notin Y$ and therefore for $x = 1$ there is no $y \in Y$ such that $x + y = 13$.

6. Yesterday, Alice, Bob, and Charlie raced against each other. The three racers placed first, second, and third (last).

   - Alice claims to have arrived first;
   - Bob claims that he did not arrive last; and
   - Charlie claims that Alice arrived before Bob.

   (a) In what order they finished the race if all of them are telling the truth?

   Answer: Since Alice is telling the truth, it follows that she finished first. Since Bob is telling the truth, he cannot finished last and therefore he finished second. Hence, Charlie finished third. Moreover, Charlie is telling the truth since Alice finished before Bob.

   (b) Could all of them be lying?

   Answer: If Bob is lying he must finished last. But then Charlie is telling the truth regardless of who finished first and who finished second because in both cases Alice finished before Bob. Therefore, it cannot be the case that all of them are lying.

   (c) If exactly one of them is lying while the other two are telling the truth, in what order they finished the race and who is the liar?

   Answer:

   - If Charlie is the only liar then Bob finished before Alice and therefore Alice is also lying. Therefore, Charlie is telling the truth.
   - The only finishing order implying that Alice is lying and Charlie is telling the truth is when Alice finished second and Bob finished last. This means that Bob is also lying. But this is impossible because only one of them is lying.
   - Hence, Bob is the only liar and as a result he finished last while Alice who is telling the truth finished first and Charlie finished second. With this order, Alice and Charlie are telling the truth while Bob is lying.

Answering all three questions with a table: There are six possible ways the race can end. For each possibility, the following table states the truthfulness of the three claims made by the three racers.

Finishing order	Alice's claim	Bob's claim	Charlie's claim
(Alice,Bob,Charlie)	T	T	T
(Alice,Charlie,Bob)	T	F	T
(Bob,Alice,Charlie)	F	T	F
(Bob,Charlie,Alice)	F	T	F
(Charlie,Alice,Bob)	F	F	T
(Charlie,Bob,Alice)	F	T	F

The only row in which all claims are TRUE is the first one. Therefore, the answer to part (a) is that Alice finished first, Bob finished second, and Charlie finished last. The answer to part (b) is "NO" since none of the rows contain three FALSE claims. The only row that contains one FALSE claim and two TRUE claims is the second row. Therefore, the answer to part (c) is that Alice finished first, Charlie finished second, and Bob finished last.

Note that the case in which exactly one of them is telling the truth while the other two are lying is inconclusive. This is because the table contains four rows with exactly one TRUE claim and it is impossible to determine the finishing order or even the identities of who finished first, or who finished second, or who finished last.