CISC 2210 – Introduction to Discrete Structures

Midterm 1 Exam Solutions

October 6, 2022

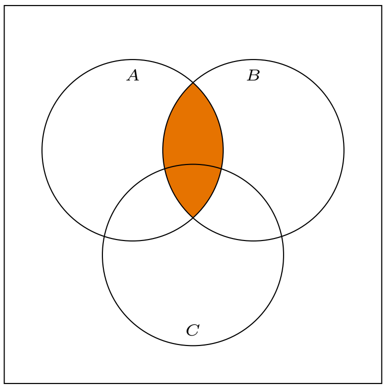
1. **Theorem:** $(A \cap B) \setminus C \equiv (A \setminus C) \cap (B \setminus C)$

   Proof: Let $L = (A \cap B) \setminus C$ and $R = (A \setminus C) \cap (B \setminus C)$. Consider the following two cases:

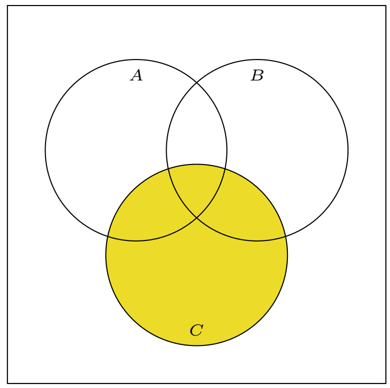
   - $x \in L$: By definition, $x \in A \cap B$ but $x \notin C$. Consequently, $x \in A \setminus C$ and $x \in B \setminus C$. It follows that $x \in R$.
   - $y \in R$: By definition, $y \in A \setminus C$ and $y \in B \setminus C$. Therefore, $y \in A \cap B$ but $y \notin C$. It follows that $y \in L$.

   The above two cases show that every object of L belongs to R and every object of R belongs to L. As a result $L \equiv R$.

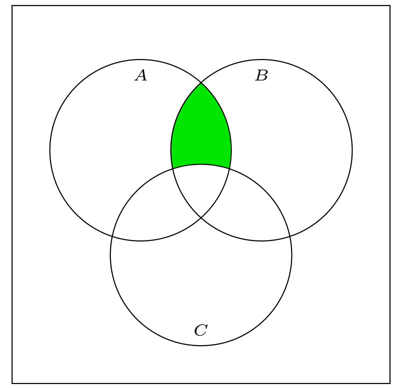
   Proof with Venn Diagrams:



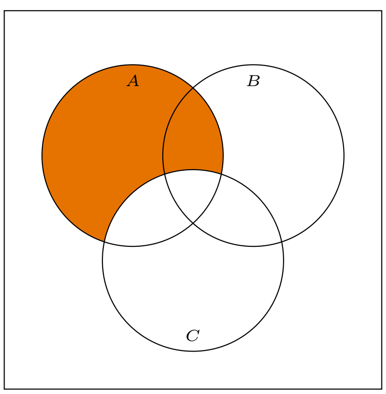
$A \cap B$



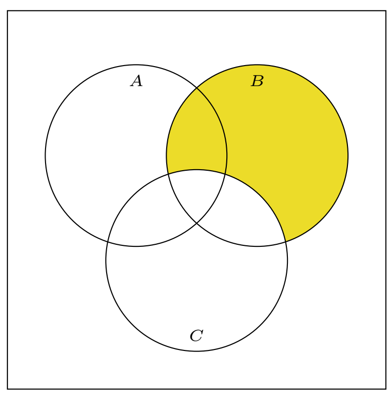
C



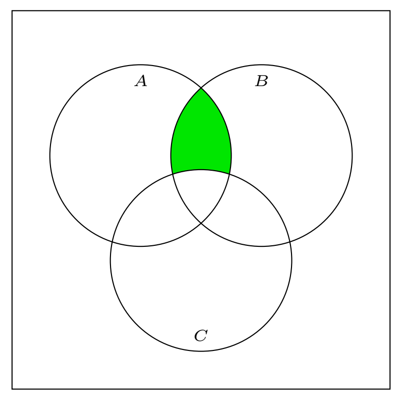
$(A \cap B) \setminus C$



$A \setminus C$



$B \setminus C$

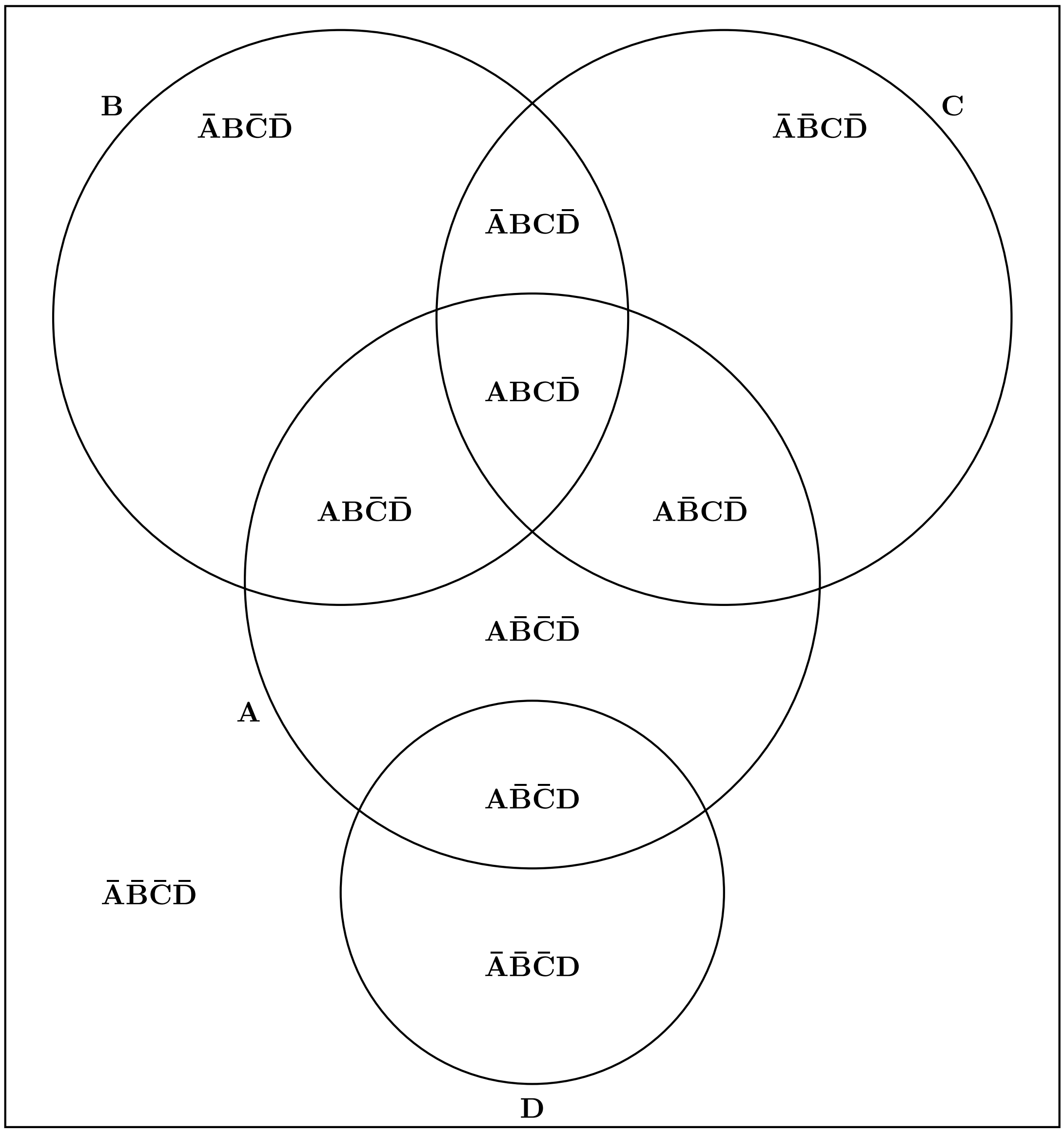


$(A \setminus C) \cap (B \setminus C)$

2. For any set S, denote by $\bar{S}$ the complement $\neg S$ of the set S. For any two sets S and T, denote by ST the intersection $S \cap T$ between the two sets.

   Example: The intersection $A \cap \neg B \cap \neg C \cap D$ is denoted by $A\bar{B}\bar{C}D$.

   Below is a Venn Diagram for some sets A, B, C, and D in which the ten existing zones are marked.



   The zones that do not appear in the diagram are all the zones that require showing an intersection between the set D with either the set B or with the set C. The six such zones are:

$$ABCD \quad AB\bar{C}D \quad A\bar{B}CD \quad \bar{A}BCD \quad \bar{A}B\bar{C}D \quad \bar{A}\bar{B}CD$$

   Note that the markings of the above six zones do not contain $\bar{D}$ and do not contain both $\bar{B}$ and $\bar{C}$.

3. The Computer Science department has **90** students: **40** students registered for the programming club, **38** students registered for the systems club, and **34** students registered for the math club.

   13 students registered for both the programming club and the systems club, **18** students registered for both the programming club and the math club, **8** students registered for both the systems club and the math club, while **12** students registered for none of the clubs.

   (a) How many students registered for all three clubs?

   (b) How many students registered only for the programming club?

   (c) How many students registered only for the system club?

   (d) How many students registered only for the math club?

Processing the input: Let P be the set of students who registered for the programming club, let S be the set of students who registered for the systems club, and let M be the set of students who registered for the math club. The provided numbers imply the following:

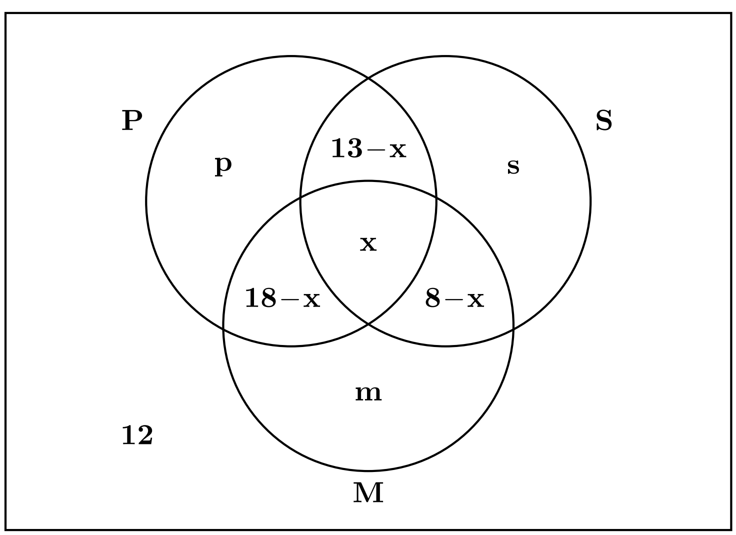
- $|P| = 40$, $|S| = 38$, and $|M| = 34$.
- $|P \cap S| = 13$, $|P \cap M| = 18$, and $|S \cap M| = 8$.
- $|\overline{P \cup S \cup M}| = 12$.

Since the number of students in the Computer Science department is 90, it follows that

$$
\begin{aligned}
|P \cup S \cup M| &= 90 - |\overline{P \cup S \cup M}| \\
&= 90 - 12 \\
&= 78
\end{aligned}
$$

Denote by $x = |P \cap S \cap M|$ the number of students who registered for all three clubs (the answer to part (a)), by $p = |P \setminus (S \cup M)|$ the number of students who registered only for the programming club (the answer to part (b)), by $s = |S \setminus (P \cup M)|$ the number of students who registered only for the systems club (the answer to part (c), and by $m = |M \setminus (P \cup S)|$ the number of students who registered only for the math club (the answer to part (d).

The Venn Diagram below shows the number of students in each of the possible 8 zones using the variables x, p, s, and m.

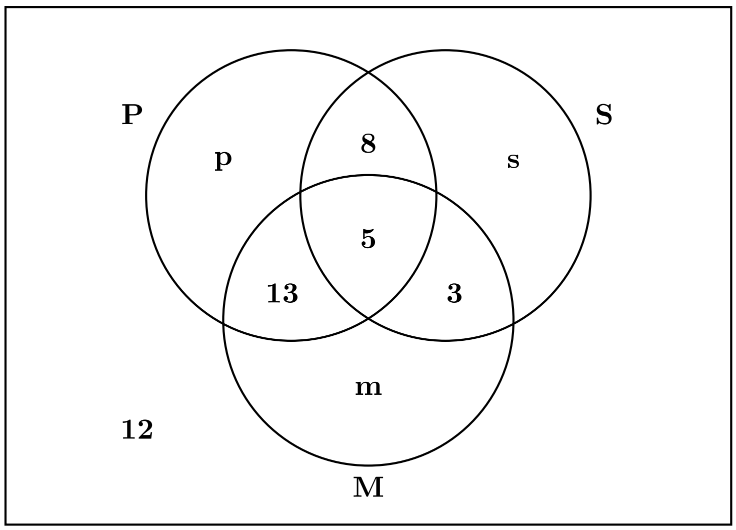


Solution to part (a): By the principle of inclusion exclusion

$$\begin{aligned} 78 &= |P \cup S \cup M| \\ &= |P| + |S| + |M| - |P \cap S| - |P \cap M| - |S \cap M| + |P \cap S \cap M| \\ &= 40 + 38 + 34 - 13 - 18 - 8 + x \\ &= 73 + x \end{aligned}$$

As a result, $x = 78 - 73 = 5$.

After substituting x with 5 in the Venn Diagram, the numbers in the 8 zones are as follows.

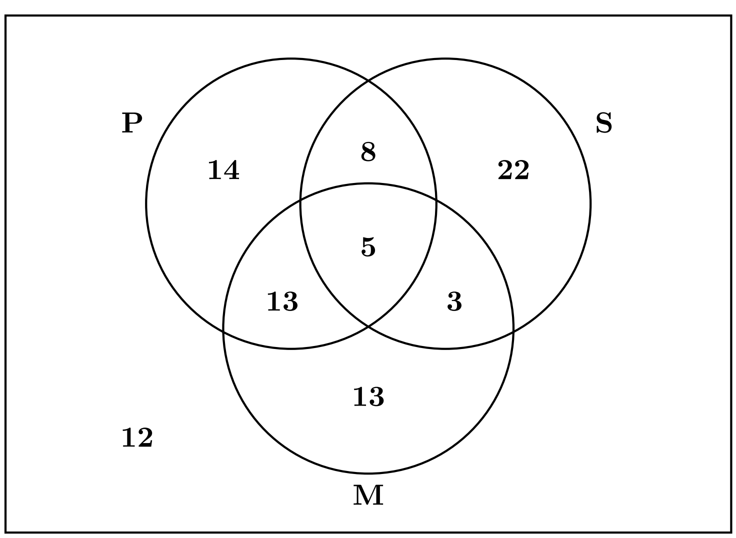


Solution to part (b): Since $|P| = 40$, it follows that $p = 40-8-13-5 = 40-26 = 14$ students registered only for the programming club.

Solution to part (c): Since $|S| = 38$, it follows that $s = 38-8-3-5 = 38-16 = 22$ students registered only for the systems club.

Solution to part (d): Since $|M| = 34$, it follows that $m = 34-13-3-5 = 34-21 = 13$ students registered only for the math club.

The Venn Diagram below shows the final numbers in all 8 zones.



4. Define a boolean formula $\mathcal{P}$ on the variables x, y, and z for which the following table is its truth table.

Optimization goal: Find as short as possible formula.

x	y	z	$\mathcal{P}$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

Answer: The table shows that $\mathcal{P}$ is TRUE only if $x = T$. However if both $y = F$ and $z = F$ then $\mathcal{P}$ is FALSE even if $x = T$. That is, $P \equiv x \wedge (y \vee z)$. See the truth table below.

x	y	z	$y \vee z$	$x \wedge (y \vee z)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Remark: One of the distributive laws for the functions $\mathcal{AND}$ ($\wedge$) and $\mathcal{OR}$ ($\vee$) is

$$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$$

Therefore $(x \wedge y) \vee (x \wedge z)$ is also a formula for which the input table is its truth table. However, the formula $x \wedge (y \vee z)$ is shorter.

5. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 4, 9, 16, 25\}$, and $Z = \{1, 8, 27, 64, 125\}$.

(a) The expression $\forall_{x \in X} \exists_{y \in Y} \exists_{z \in Z} \{x \times y = z\}$ is **TRUE**.

Proof: Observe that X contains the first five positive integers, Y contains the five squares of these integers, and Z contains the five cubes of these integers. As a result, for any selection of $x \in X$, select $y = x^2$ and $z = x^3$ to get $x \times y = x \times x^2 = x^3 = z$.

Specifically, for $x = 1$ select $y = 1$ and $z = 1$, for $x = 2$ select $y = 4$ and $z = 8$, for $x = 3$ select $y = 9$ and $z = 27$, for $x = 4$ select $y = 16$ and $z = 64$, and for $x = 5$ select $y = 25$ and $z = 125$. In all five cases $x \times y = z$.

Remark: Note that for any selection of $x \in X$, these are the only possible selections of $y \in Y$ and $z \in Z$ proving that the expression is TRUE.

(b) The expression $\exists_{x \in X} \exists_{y \in Y} \forall_{z \in Z} \{x \times y \geq z\}$ is **TRUE**.

Proof: Observe that since the product of x and y is required to be greater than or equal to z, the best strategy to prove that the expression is TRUE is to select the largest possible values for x and y. That is, select $x = 5$ and $y = 25$ to get $x \times y = 125$ which is greater than or equal to z for all $z \in Z$.

Remark: Note that because all other possible $x \times y$ products are less than 125, it follows that $x = 5$ and $y = 25$ are the only possible selections of $x \in X$ and $y \in Y$ proving that the expression is TRUE.

(c) The expression $\forall_{x \in X} \exists_{y \in Y} \forall_{z \in Z} \{x \times y \leq z\}$ is **FALSE**.

Proof: Observe that since the product of x and y is required to be less than or equal to z, the best strategy to find a counterexample is to select the largest possible value for x and the smallest possible value for z. That is, select $x = 5$ and $z = 1$. Now, $x \times y = 5 \times y \geq 5 > z$ for all $y \in Y$. This counterexample implies that the expression is FALSE.

Remark: Note that there are more counterexamples. Any selection of $x > 1$ with $z = 1$ implies a counterexample. On the other hand, for any selection of $z > 1$, it follows that for all $x \in X$ the selection of $y = 1$ implies that $x \times y \leq z$.

6. Alice claims that today is Sunday. Bob claims that it is rainy outside. Charlie claims that if Alice is telling the truth then Bob is lying. **Charlie is lying.** What day is it today and is it rainy outside?

   Answer 1 with logical arguments: If Alice is lying then Charlie is not lying because he claimed nothing about what happens if Alice is lying. Therefore, Alice is telling the truth. If Alice is telling the truth and Bob is lying then Charlie is telling the truth. Since Charlie is lying, it must be the case that both Alice and Bob are telling the truth. Therefore, today is Sunday and it is rainy outside.

   Answer 2 with propositional logic: Let P be the boolean proposition: "Alice is telling the truth" and let Q be the boolean proposition "Bob is lying". As a result, Charlie's claim can be expressed with the boolean proposition: $P \to Q$. It follows that if Charlie is lying then $P \to Q$ must be FALSE. By the definition of the function $\mathcal{IMPLY}$, the proposition $P \to Q$ is FALSE only if P is TRUE and Q is FALSE. Thus, both Alice and Bob are telling the truth. Therefore, today is Sunday and it is rainy outside.