CISC 2210 (TR11) – Introduction to Discrete Structures

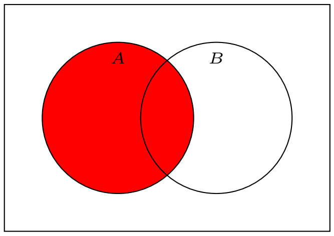
Midterm 1 Exam – Solutions

September 26, 2023

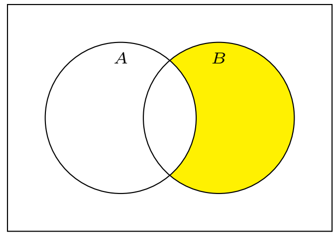
1. Let A and B be two non-empty sets. Find the simplest form for each of the following four expressions involving these sets.

   (a) What is $A \cap (B \setminus A)$?

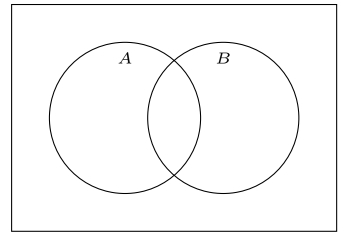
   Answer: $(B \setminus A)$ does not contain objects from A. Therefore, the intersection between A and $(B \setminus A)$ must be empty. That is, $\mathbf{A} \cap (\mathbf{B} \setminus \mathbf{A}) = \emptyset$.



A



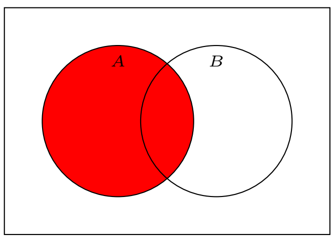
$B \setminus A$



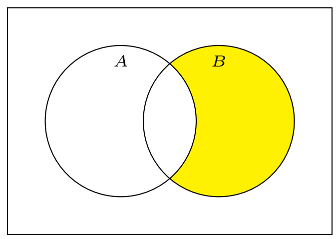
$\mathbf{A} \cap (\mathbf{B} \setminus \mathbf{A}) = \emptyset$

   (b) What is $A \cup (B \setminus A)$?

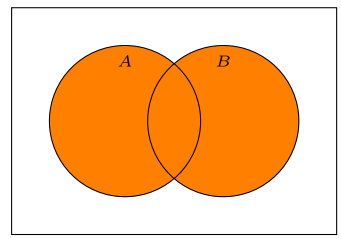
   Answer: $(B \setminus A)$ contains all the object of B that are not in A. Therefore, the union of A and $(B \setminus A)$ is also the union of A and B. That is, $\mathbf{A} \cup (\mathbf{B} \setminus \mathbf{A}) = \mathbf{A} \cup \mathbf{B}$.



A



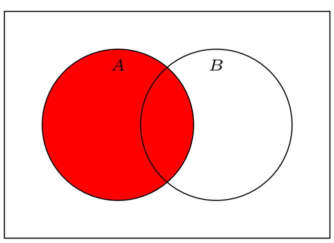
$B \setminus A$



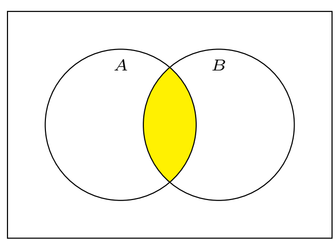
$\mathbf{A} \cup (\mathbf{B} \setminus \mathbf{A}) = \mathbf{A} \cup \mathbf{B}$

   (c) What is $A \setminus (A \cap B)$?

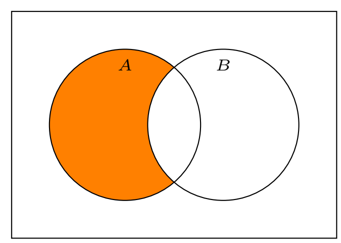
   Answer: $A \cap B$ is a subset of A that contains all the objects from A that are also in B. Therefore, after removing this intersection from A what is left is $(A \setminus B)$. That is, $\mathbf{A} \setminus (\mathbf{A} \cap \mathbf{B}) = \mathbf{A} \setminus \mathbf{B}$.



A



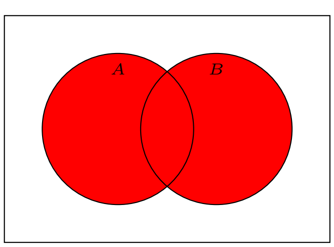
$A \cap B$



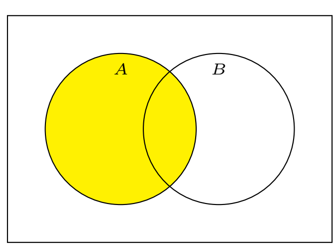
$\mathbf{A} \setminus (\mathbf{A} \cap \mathbf{B}) = \mathbf{A} \setminus \mathbf{B}$

   (d) What is $(A \cup B) \setminus A$?

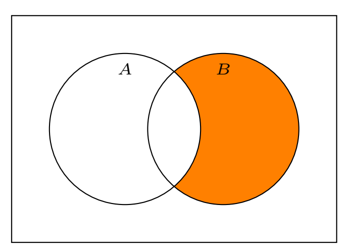
   Answer: $(A \cup B)$ contains all the objects from both A and B. Therefore, after removing from $(A \cup B)$ all the objects from A what remain are the objects from B that are not in A. That is, $(\mathbf{A} \cup \mathbf{B}) \setminus \mathbf{A} = \mathbf{B} \setminus \mathbf{A}$.



$A \cup B$



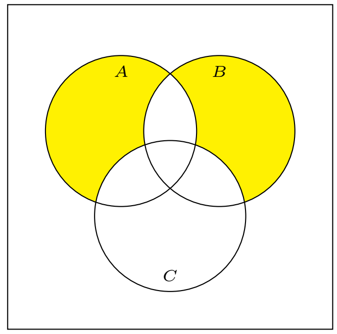
A



$(\mathbf{A} \cup \mathbf{B}) \setminus \mathbf{A} = \mathbf{B} \setminus \mathbf{A}$

2. (a) In the following Venn Diagram for the three sets A, B, and C, mark the zones that must be empty if $\mathbf{A} \cup \mathbf{C} = \mathbf{B} \cup \mathbf{C}$.

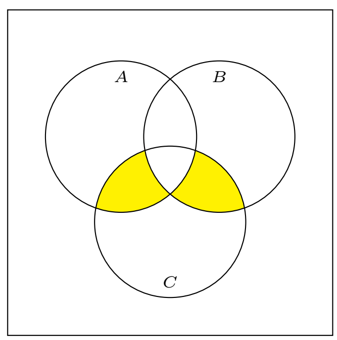
Answer: A cannot contain objects that belong only to A because such objects belong to $A \cup C$ and do not belong to $B \cup C$. Similarly, B cannot contain objects that belong only to B because such objects belong to $B \cup C$ and do not belong to $A \cup C$. Therefore, the zones $A \setminus (B \cup C)$ and $B \setminus (A \cup C)$ must be empty. These zones are marked yellow in the diagram below.



$(A \setminus (B \cup C)) \cup (B \setminus (A \cup C))$

(b) In the following Venn Diagram for the three sets A, B, and C, mark the zones that must be empty if $\mathbf{A} \cap \mathbf{C} = \mathbf{B} \cap \mathbf{C}$.

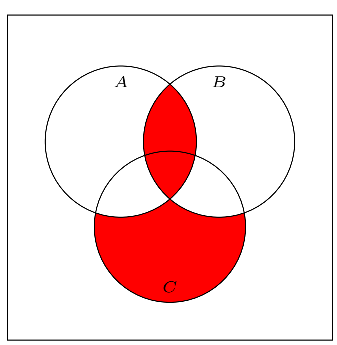
Answer: $A \cap C$ cannot contain objects that do not belong to B and $B \cap C$ cannot contain objects that do not belong to A because otherwise $A \cap C \neq B \cap C$. Therefore, the zones $(A \cap C) \setminus B$ and $(B \cap C) \setminus A$ must be empty. These zones are marked yellow in the diagram below.



$((A \cap C) \setminus B) \cup ((B \cap C) \setminus A)$

(c) What is the relationship between the sets A and B if both $\mathbf{A} \cup \mathbf{C} = \mathbf{B} \cup \mathbf{C}$ and $\mathbf{A} \cap \mathbf{C} = \mathbf{B} \cap \mathbf{C}$?

Answer: It must be the case that $\mathbf{A} = \mathbf{B}$. The red zones in the diagram below are the complement of the union of all the yellow zones in the two diagrams above. These red zones demonstrate that if both equalities hold then both $A \setminus B$ and $B \setminus A$ must be empty and therefore $A \cap B = A = B$.



$(A \cap B) \cup (C \setminus (A \cup B))$

3. All of the students of the Brooklyn College CIS department won a free one week vacation in Europe to visit one or two out of the following four countries: England (**E**), France (**F**), Germany (**G**), and Italy (**I**). Students were not allowed to tour more than two countries.

   It so happened that among the students who took the tour to Europe, each one of the four countries was toured by exactly **40** students while each possible pair of countries was toured by exactly **10** students.

   Remark: There are six possible pairs of countries: {**E,F**}, {**E,G**}, {**E,I**}, {**F,G**}, {**F,I**}, and {**G,I**}.

   (a) How many CIS students took the free vacation to Europe?

   Answer: **100** students.

   Proof: Let E denote the set of students who toured England, let F denote the set of students who toured France, let G denote the set of students who toured Germany, and let I denote the set of students who toured Italy. The goal is to find the number of students who took the vacation to Europe. That is, the goal is to find the size of the set

$$E \cup F \cup G \cup I$$

   Observe that since students may tour at most two countries, it follows that all the intersections among three or four sets out of the four sets E, F, G, and I must be empty. As a result, the principle of inclusion exclusion for these four sets becomes

$$|E \cup F \cup G \cup I| = |E| + |F| + |G| + |I| - |E \cap F| - |E \cap G| - |E \cap I| - |F \cap G| - |F \cap I| - |G \cap I|$$

   The given sizes of the four sets and the six possible pairwise intersections among them imply that

$$|E \cup F \cup G \cup I| = 40 + 40 + 40 + 40 - 10 - 10 - 10 - 10 - 10 - 10 = 160 - 60 = \mathbf{100}$$

   (b) How many CIS students toured only England?

   Answer: **10** students.

   Proof: Let X denote the set of students who toured only England. The goal is to find the size of X. By definition, the set E is the union of four of its subsets X, $E \cap F$, $E \cap G$, and $E \cap I$. That is,

$$E = X \cup (E \cap F) \cup (E \cap G) \cup (E \cap I)$$

   Moreover, these four subsets of E are disjoint because students are allowed to tour at most two countries. As a result,

$$|E| = |X| + |E \cap F| + |E \cap G| + |E \cap I|$$

   The given sizes of E and the three possible pairwise intersections with E imply that

$$40 = |X| + 10 + 10 + 10 = |X| + 30$$

   It follows that

$$|X| = 40 - 30 = \mathbf{10}$$

4. The following is the truth table of the function $\mathcal{XOR}$ (denoted by $\oplus$) with the boolean variables x and y.

x	y	$x \oplus y$
T	T	F
T	F	T
F	T	T
F	F	F

Prove that the following proposition with the three boolean variables x, y and z is a contradiction (that is, there is no truth assignment to this proposition):

$$(x \oplus y) \wedge (y \oplus z) \wedge (z \oplus x)$$

Notations: Denote the proposition by P and define $P_1 = x \oplus y$, $P_2 = y \oplus z$, and $P_3 = z \oplus x$. That is,

$$P = P_1 \wedge P_2 \wedge P_3$$

Proof I: Since P is the $\mathcal{AND}$ of P_1, P_2, and P_3, it follows that $P = F$ if at least one of P_1, P_2, and P_3 is F. In particular, if $P_1 = F$ then $P = F$. Otherwise, assume that $P_1 = T$. There are two cases:

(a) $x = T$ and $y = F$

(b) $x = F$ and $y = T$

In the first case, if $z = T$ then $P_3 = (z \oplus x) = F$ and if $z = F$ then $P_2 = (y \oplus z) = F$. In both sub-cases $P = F$. In the second case, if $z = T$ then $P_2 = (y \oplus z) = F$ and if $z = F$ then $P_3 = (z \oplus x) = F$. In both sub-cases $P = F$.

Proof II: In any truth assignment to x, y, and z, it must be the case that two of them get the same truth assignment either both of them are T or both of them are F. In either case, the $\mathcal{XOR}$ of both of them, either P_1, or P_2, or P_3, is F. As a result $P = F$ because it is the $\mathcal{AND}$ of P_1, P_2, and P_3.

A proof with a truth table: The proposition is a contradiction because the last column in the following table is all F.

x	y	z	$x \oplus y$	$y \oplus z$	$z \oplus x$	$(x \oplus y) \wedge (y \oplus z) \wedge (z \oplus x)$
T	T	T	F	F	F	F
T	T	F	F	T	T	F
T	F	T	T	T	F	F
T	F	F	T	F	T	F
F	T	T	T	F	T	F
F	T	F	T	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

5. Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of all 10 digits.

   The following eight boolean propositions include universal quantifiers, existential quantifiers, or their negations ($\forall$ or $\neg(\forall)$ or $\exists$ or $\neg(\exists)$).

   Which **four** of them are TRUE and which **four** of them are FALSE?

   (a) $\forall_{x \in D} \forall_{y \in D} \, (x < y)$

   Answer: FALSE.

   Explanation I: The proposition is false for many pairs of x and y. For example, $x = 9$ and $y = 0$.

   Explanation II: This proposition is false because the proposition in part (c) is false.

   (b) $\forall_{x \in D} \exists_{y \in D} \, (x < y)$

   Answer: FALSE.

   Explanation: The proposition is false for $x = 9$ because there is no y that could be greater than 9. Note, that the proposition is true for $x < 9$ because of $y = 9$.

   (c) $\exists_{x \in D} \forall_{y \in D} \, (x < y)$

   Answer: FALSE.

   Explanation: The proposition is false for any value of x because $x \not< y$ when $y = 0$ or when $y = x$.

   (d) $\exists_{x \in D} \exists_{y \in D} \, (x < y)$

   Answer: TRUE.

   Explanation: There are many pairs of x and y that make the proposition true. For example, $x = 0$ and $y = 9$.

   (e) $\forall_{x \in D} \neg(\forall_{y \in D} \, (x < y))$

   Answer: TRUE.

   Explanation: For any value of x, the expression $\forall_{y \in D} \, (x < y)$ is false because $x \not< y$ when $y = 0$ or when $y = x$. Therefore, $\neg(\forall_{y \in D} \, (x < y))$ the negation of this expression is true for all possible values of x. This implies that the proposition is true.

   (f) $\forall_{x \in D} \neg(\exists_{y \in D} \, (x < y))$

   Answer: FALSE.

   Explanation: For $x < 9$, the expression $\exists_{y \in D} \, (x < y)$ is true for $y = 9$. Therefore, $\neg(\exists_{y \in D} \, (x < y))$ the negation of this expression is false for all values of x except for $x = 9$. This implies that the proposition is false.

   (g) $\exists_{x \in D} \neg(\forall_{y \in D} \, (x < y))$

   Answer: TRUE.

   Explanation I: For any value of x, the expression $\forall_{y \in D} \, (x < y)$ is false because $x \not< y$ when $y = 0$ or when $y = x$. Therefore, $\neg(\forall_{y \in D} \, (x < y))$ the negation of this expression is true. This implies that the proposition is true for all possible values of x.

   Explanation II: This proposition is true because the proposition in part (e) is true.

   (h) $\exists_{x \in D} \neg(\exists_{y \in D} \, (x < y))$

   Answer: TRUE.

   Explanation: For $x = 9$, the expression $\exists_{y \in D} \, (x < y)$ is false because y cannot be greater than 9. Therefore, $\neg(\exists_{y \in D} \, (x < y))$ the negation of this expression is true. This implies that the proposition is true for $x = 9$. Note, that the proposition is false for $x < 9$ because of $y = 9$.

6. (a) A drawer contains **10** white socks and **10** black socks. Without looking, you blindly draw random socks from the drawer one at a time.

   - What is the minimum number of socks you need to draw to guarantee that you have a pair of matching color socks?

     Answer: 3.

     Explanation: If you are lucky, then after drawing 2 socks, both of them are white or both of them are black. If you are not lucky, then one of them is white and one of them is black. However, in this case, the third sock guarantees a match beacuse it is either white or black.

   - What is the answer if the drawer contains **100** white socks and **100** black socks?

     Answer: 3.

     Explanation: Exactly the same as in the case of 10 white socks and 10 black socks.

   Generalization I: Assume that the drawer contains $w \geq 2$ white socks and $b \geq 2$ black socks. Then after drawing 3 socks you are guaranteed to have 2 socks of the same color.

   Generalization II: For $n \geq 2$, assume that the drawer contains n white socks, n black socks, and n gray socks. Then after drawing 4 socks, you are guaranteed to have at least 2 socks of the same color.

   Challenge: Assume that there are socks of k colors in the drawer. Call these colors $c_1, c_2, \ldots, c_k$. Also assume that for $1 \leq i \leq k$, the drawer contains $n_i \geq 2$ socks of color c_i. What is the minimum number of socks you need to draw to guarantee that you have two socks of the same color?

   (b) A drawer contains **10** identical pairs of black shoes (a right shoe and a left shoe). Without looking, you blindly draw random shoes from the drawer one at a time.

   - What is the minimum number of shoes you need to draw to guarantee that you have at least one right shoe and at least one left shoe?

     Answer: 11.

     Explanation: If you are lucky, then after drawing 2 shoes, you will have one right shoe and one left shoe. However, even 10 shoes are not enough to guarantee drawing at least one right shoe and at least one left shoe. This is because all of them might be right shoes or all of them might be left shoes. In the first case, the next shoe must be a left shoe and in the second case, the next shoe must be a right shoe. This is because there are only 10 right shoes and only 10 left shoes. In both cases, after you draw the 11^{th} shoe, you will have at least one right shoe and at least one left shoe.

   - What is the answer if the drawer contains **100** identical pairs of black shoes?

     Answer: 101.

     Explanation: The explanation is the same as in the case with 10 identical pairs. Just replace 10 with 100 and 11 with 101.

   Generalization: Assume that the drawer contains $r \geq 1$ right shoes and $\ell \geq 1$ left shoes. Then after drawing $\max\{r, \ell\} + 1$ shoes, you are guaranteed to have at least one right shoe and at least one left shoe. In particular, if $r = \ell = n$, then after drawing $n + 1$ shoes you are guaranteed to have at least one right shoe and at least one left shoe.