CISC 2210 (TR11) – Introduction to Discrete Structures

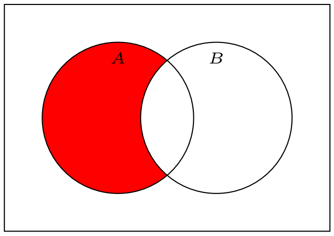
Midterm 1 Exam – Solutions

September 26, 2023

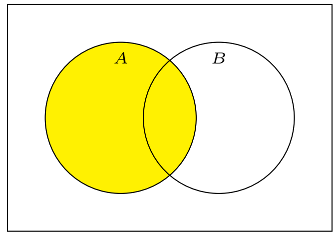
1. Let A and B be two non-empty sets. Find the simplest form for each of the following four expressions involving these sets.

   (a) What is $(A \setminus B) \setminus A$?

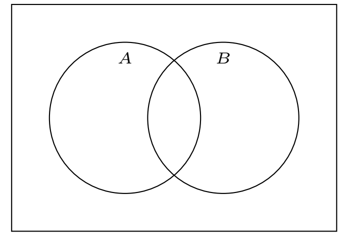
   Answer: $(A \setminus B)$ is a subset of A. This implies that after removing from $(A \setminus B)$ all the objects that belong to A what remains is an empty set. That is, $(\mathbf{A} \setminus \mathbf{B}) \setminus \mathbf{A} = \emptyset$.



$A \setminus B$



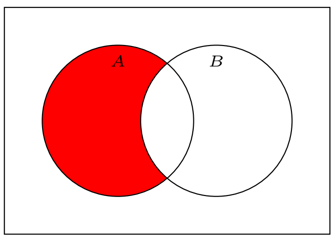
A



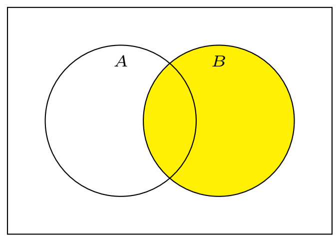
$(\mathbf{A} \setminus \mathbf{B}) \setminus \mathbf{A} = \emptyset$

   (b) What is $(A \setminus B) \setminus B$?

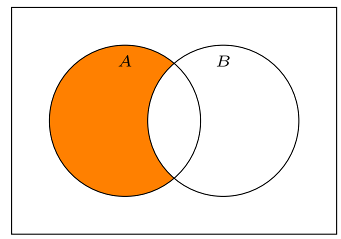
   Answer: $(A \setminus B)$ does not contain objects from B. This implies that removing from $(A \setminus B)$ objects that belong to B does not change $A \setminus B$. That is, $(\mathbf{A} \setminus \mathbf{B}) \setminus \mathbf{B} = \mathbf{A} \setminus \mathbf{B}$.



$A \setminus B$



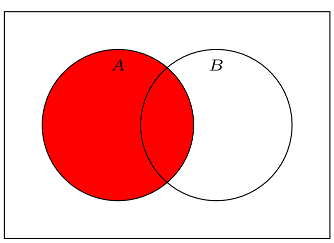
B



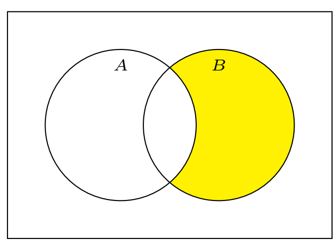
$(\mathbf{A} \setminus \mathbf{B}) \setminus \mathbf{B} = \mathbf{A} \setminus \mathbf{B}$

   (c) What is $A \setminus (B \setminus A)$?

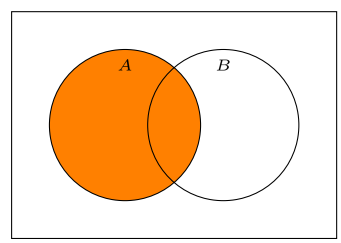
   Answer: $(B \setminus A)$ does not contain objects from A. This implies that removing from A objects that belong to $(B \setminus A)$ does not change A. That is, $\mathbf{A} \setminus (\mathbf{B} \setminus \mathbf{A}) = \mathbf{A}$.



A



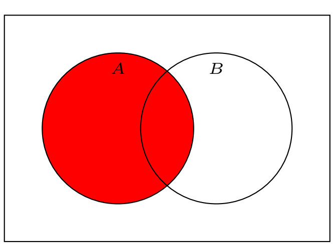
$B \setminus A$



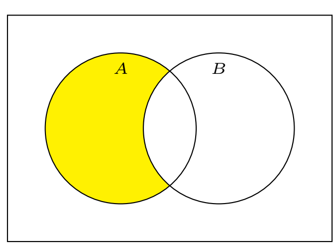
$\mathbf{A} \setminus (\mathbf{B} \setminus \mathbf{A}) = \mathbf{A}$

   (d) What is $A \setminus (A \setminus B)$?

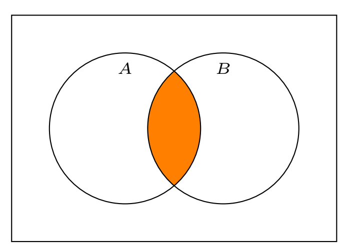
   Answer: $(A \setminus B)$ is a subset of A that contains all the objects from A that do not belong to B. this implies that after removing from A all the objects that belong to $(A \setminus B)$ what remains are the objects in $(A \cap B)$. That is, $\mathbf{A} \setminus (\mathbf{A} \setminus \mathbf{B}) = \mathbf{A} \cap \mathbf{B}$.



A



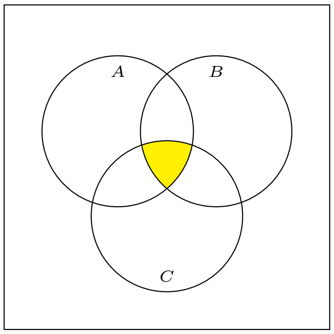
$A \setminus B$



$\mathbf{A} \setminus (\mathbf{A} \setminus \mathbf{B}) = \mathbf{A} \cap \mathbf{B}$

2. (a) In the following Venn Diagram for the three sets A, B, and C, mark the zone that must be empty if $(\mathbf{A} \cap \mathbf{C}) \cap (\mathbf{B} \cap \mathbf{C}) = \emptyset$.

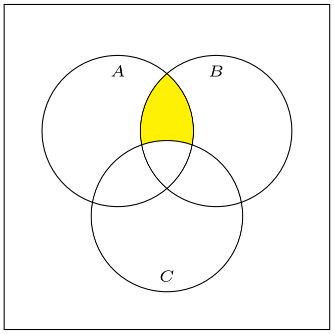
Answer: Observe that $(A \cap C) \cap (B \cap C) = A \cap B \cap C$. Therefore the intersection among the three sets must be empty. The zone that represents this intersection is marked yellow in the diagram below.



$A \cap B \cap C$

(b) In the following Venn Diagram for the three sets A, B, and C, mark the zone that must be empty if $(\mathbf{A} \setminus \mathbf{C}) \cap (\mathbf{B} \setminus \mathbf{C}) = \emptyset$.

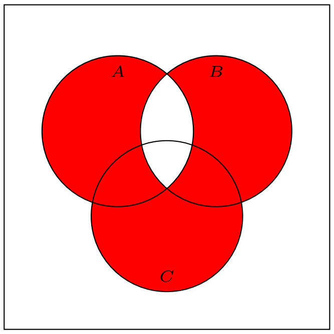
Answer: Observe that $(A \setminus C) \cap (B \setminus C) = (A \cap B) \setminus C$. Therefore the intersection between A and B that does not contain object from C must be empty. The zone that represents this subset of the intersection is marked yellow in the diagram below.



$(A \cap B) \setminus C$

What can you say about the intersection $A \cap B$ if both $(\mathbf{A} \cap \mathbf{C}) \cap (\mathbf{B} \cap \mathbf{C}) = \emptyset$ and $(\mathbf{A} \setminus \mathbf{C}) \cap (\mathbf{B} \setminus \mathbf{C}) = \emptyset$?

Answer: It must be the case that $\mathbf{A} \cap \mathbf{B} = \emptyset$. The red zones in the diagram below are the complement of the union of the two yellow zones in the two diagrams above. These red zones demonstrate that if both equalities hold then $A \cap B$ must be empty.



$(A \cap B) \cup (C \setminus (A \cup B))$

3. Many of the Brooklyn College students love playing a variety of sports. Each year, the college offers each student free practice sessions in at most two of the following four sports: Basketball (**B**), Soccer (**S**), Tennis (**T**), and Volleyball (**V**). Students are not allowed to select more than two sports.

   It so happened that among the students who chose to participate in the free practice sessions in 2023, each one of the four sports was selected by exactly **100** students while each possible pair of sports was selected by exactly **20** students.

   Remark: There are six possible pairs of sportss: {**B,S**}, {**B,T**}, {**B,V**}, {**S,T**}, {**S,V**}, and {**T,V**}.

   (a) In 2023, how many students participated in at least one free practice session?

   Answer: **280** students.

   Proof: Let B denote the set of students who participated in the basketball practice session, let S denote the set of students who participated in the soccer practice session, let T denote the set of students who participated in the Tennis practice session, and let V denote the set of students who participated in the volleyball practice session. The goal is to find the number of students who participated in at least one practice session. That is, the goal is to find the size of the set

$$B \cup S \cup T \cup V$$

   Observe that since students may choose to practice in at most two practice sessions, it follows that all the intersections among three or four sets out of the four sets B, S, T, and V must be empty. As a result, the principle of inclusion exclusion for these four sets becomes

$$|B \cup S \cup T \cup V| = |B| + |S| + |T| + |V| - |B \cap S| - |B \cap T| - |B \cap V| - |S \cap T| - |S \cap V| - |T \cap V|$$

   The given sizes of the four sets and the six possible pairwise intersections among them imply that

$$|B \cup S \cup T \cup V| = 100 + 100 + 100 + 100 - 20 - 20 - 20 - 20 - 20 - 20 = 400 - 120 = \mathbf{280}$$

   (b) In 2023, how many students participated only in the Tennis practice session?

   Answer: **40** students.

   Proof: Let X denote the set of students who participated only in the Tennis practice session. The goal is to find the size of X. By definition, the set T is the union of four of its subsets X, $T \cap B$, $T \cap S$, and $T \cap V$. That is,

$$T = X \cup (T \cap B) \cup (T \cap S) \cup (T \cap V)$$

   Moreover, these four subsets of T are disjoint because students are allowed to choose at most two practice sessions. As a result,

$$|T| = |X| + |T \cap B| + |T \cap S| + |T \cap V|$$

   The given sizes of T and the three possible pairwise intersections with T imply that

$$100 = |X| + 20 + 20 + 20 = |X| + 60$$

   It follows that

$$|X| = 100 - 60 = \mathbf{40}$$

4. The following is the truth table of the function $\mathcal{EQUIV}$ (denoted by $\equiv$) with the boolean variables x and y.

x	y	$x \equiv y$
T	T	T
T	F	F
F	T	F
F	F	T

Prove that the following proposition with the three boolean variables x, y and z is a tautology (that is, all the eight possible assignments satisfy this proposition):

$$(x \equiv y) \vee (y \equiv z) \vee (z \equiv x)$$

Notations: Denote the proposition by P and define $P_1 = (x \equiv y)$, $P_2 = (y \equiv z)$, and $P_3 = (z \equiv x)$. That is,

$$P = P_1 \vee P_2 \vee P_3$$

Proof I: Since P is the $\mathcal{OR}$ of P_1, P_2, and P_3, it follows that $P = T$ if at least one of P_1, P_2, and P_3 is T. In particular, if $P_1 = T$ then $P = T$. Otherwise, assume that $P_1 = F$. There are two cases:

(a) $x = T$ and $y = F$

(b) $x = F$ and $y = T$

In the first case, if $z = T$ then $P_3 = (z \equiv x) = T$ and if $z = F$ then $P_2 = (y \equiv z) = T$. In both sub-cases $P = T$. In the second case, if $z = T$ then $P_2 = (y \equiv z) = T$ and if $z = F$ then $P_3 = (z \equiv x) = T$. In both sub-cases $P = T$.

Proof II: In any truth assignment to x, y, and z, it must be the case that two of them get the same truth assignment either both of them are T or both of them are F. In either case, the $\mathcal{EQUIV}$ of both of them, either P_1, or P_2, or P_3, is T. As a result $P = T$ because it is the $\mathcal{OR}$ of P_1, P_2, and P_3.

A proof with a truth table: The proposition is a tautology because the last column in the following table is all T.

x	y	z	$x \equiv y$	$y \equiv z$	$z \equiv x$	$(x \equiv y) \vee (y \equiv z) \vee (z \equiv x)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	F	T	F	T
F	T	F	F	F	T	T
F	F	T	T	F	F	T
F	F	F	T	T	T	T

5. Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of all 10 digits.

   The following eight boolean propositions include universal quantifiers, existential quantifiers, or their negations ($\forall$ or $\neg(\forall)$ or $\exists$ or $\neg(\exists)$).

   Which **four** of them are TRUE and which **four** of them are FALSE?

   (a) $\exists_{x \in D} \exists_{y \in D} (x > y)$

   Answer: TRUE.

   Explanation: There are many pairs of x and y that make the proposition true. For example, $x = 9$ and $y = 0$.

   (b) $\exists_{x \in D} \forall_{y \in D} (x > y)$

   Answer: FALSE.

   Explanation: The proposition is false for any value of x because $x \not> y$ when $y = 9$ or when $y = x$.

   (c) $\forall_{x \in D} \exists_{y \in D} (x > y)$

   Answer: FALSE.

   Explanation: The proposition is false for $x = 0$ because there is no y that could be smaller than 0. Note, that the proposition is true for $x > 0$ because of $y = 0$.

   (d) $\forall_{x \in D} \forall_{y \in D} (x > y)$

   Answer: FALSE.

   Explanation I: The proposition is false for many pairs of x and y. For example, $x = 0$ and $y = 9$.

   Explanation II: This proposition is false because the proposition in part (b) is false.

   (e) $\exists_{x \in D} \neg(\exists_{y \in D} (x > y))$

   Answer: TRUE.

   Explanation: For $x = 0$, the expression $\exists_{y \in D} (x > y)$ is false because y cannot be smaller than 0. Therefore, $\neg(\exists_{y \in D} (x > y))$ the negation of this expression is true. This implies that the proposition is true for $x = 0$. Note, that the proposition is false for $x > 0$ because of $y = 0$.

   (f) $\exists_{x \in D} \neg(\forall_{y \in D} (x > y))$

   Answer: TRUE.

   Explanation I: For any value of x, the expression $\forall_{y \in D} (x > y)$ is false because $x \not> y$ when $y = 9$ or when $y = x$. Therefore, $\neg(\forall_{y \in D} (x > y))$ the negation of this expression is true. This implies that the proposition is true for all possible values of x.

   Explanation II: This proposition is true because the proposition in part (h) is true.

   (g) $\forall_{x \in D} \neg(\exists_{y \in D} (x > y))$

   Answer: FALSE.

   Explanation: For $x > 0$, the expression $\exists_{y \in D} (x > y)$ is true for $y = 0$. Therefore, $\neg(\exists_{y \in D} (x < y))$ the negation of this expression is false for all values of x except for $x = 0$. This implies that the proposition is false.

   (h) $\forall_{x \in D} \neg(\forall_{y \in D} (x > y))$

   Answer: TRUE.

   Explanation: For any value of x, the expression $\forall_{y \in D} (x > y)$ is false because $x \not> y$ when $y = 9$ or when $y = x$. Therefore, $\neg(\forall_{y \in D} (x > y))$ the negation of this expression is true for all possible values of x. This implies that the proposition is true.

6. (a) A box contains **7** blue balls and **7** red balls. Without looking, you blindly draw random balls from the box one at a time.

- What is the minimum number of balls you need to draw to guarantee that you have at least one blue ball and at least one red ball?

  Answer: 8.

  Explanation: If you are lucky, then after drawing 2 balls, you will have one blue ball and one red ball. However, even 7 balls are not enough to guarantee drawing at least one ball of each color. This is because all of them might be blue or all of them might be red. In the first case, the next ball must be red and in the second case, the next ball must be blue. This is because there are only 7 balls of each color. In both cases, after you draw the 8^{th} ball, you will have at least one blue ball and at least one red ball.

- What is the answer if the box contains **25** blue balls and **25** red balls?

  Answer: 26.

  Explanation: The explanation is the same as in the case with 7 blue balls and 7 red balls. Just replace 7 with 25 and 8 with 26.

Generalization I: Assume that the box contains $b \geq 1$ blue balls and $r \geq 1$ red balls. Then after drawing $\max\{b, r\} + 1$ balls, you are guaranteed to have at least one blue ball and at least one red ball. In particular, if $b = r = n$, then after drawing $n + 1$ balls you are guaranteed to have at least one blue ball and at least one red ball.

Generalization II: For $n \geq 1$, assume that the box contains n blue balls, n red balls, and n green balls. Then after drawing $2n + 1$ balls you are guaranteed to have at least one ball of each color.

Challenge: Assume that the box contains $b \geq 1$ blue balls, $r \geq 1$ red balls, and $g \geq 1$ green balls. What is the minimum number of balls you need to draw to guarantee that you have at least one blue ball, at least one red ball, and at least one green ball?

(b) A box contains **7** blue balls and **7** red balls. Without looking, you blindly draw random balls from the box one at a time.

- What is the minimum number of balls you need to draw to guarantee that you have two balls of the same color?

  Answer: 3.

  Explanation: If you are lucky, then after drawing 2 balls, both of them are blue or both of them are red. If you are not lucky, then one of them is blue and one of them is red. However, in this case, the third ball guarantees a match beacuse it is either blue or red.

- What is the answer if the box contains **25** blue balls and **25** red balls?

  Answer: 3.

  Explanation: Exactly the same as in the case of 7 blue balls and 7 red balls.

Generalization I: Assume that the box contains $b \geq 2$ blue balls and $r \geq 2$ red balls. Then after drawing 3 balls you are guaranteed to have 2 balls of the same color.

Generalization II: For $n \geq 2$, assume that the box contains n blue balls, n red balls, and n green balls. Then after drawing 4 balls, you are guaranteed to have at least 2 balls of the same color.

Challenge: Assume that there are balls of k colors in the box. Call these colors $c_1, c_2, \ldots, c_k$. Also assume that for $1 \leq i \leq k$, the box contains $n_i \geq 2$ balls of color c_i. What is the minimum number of balls you need to draw to guarantee that you have two balls of the same color?