CISC 2210 TR11 – Introduction to Discrete Structures

Midterm 2 Exam

Nov 7, 2023

Id: ....

Problem	Maximum Points	Your Points
1	100	
2	100	
3	100	
4	100	
5	100	

Structure, problem selection, and credit:

- You have 2 hours to complete the exam.
- There are 5 problems. Each problem is a "mini-exam" by itself with a 5% weight in the final grade for the class. However, the grade of each individual problem counts only if it is higher than the final exam grade.

  Strategy: It is better to try first answering the questions relating to topics you have mastered. Note that since there is no cumulative grade, one fully correct answer is better than two or more partially correct answers.
- You will get only partial credit if you fail to justify or prove your answers. You will get 20% of the credit for any problem or part of a problem if you leave the allocated space for the answer empty. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam **by themselves** without any external help from other people, the Internet, books, notes, or calculators. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.

1. Prove the correctness of the following identity for any integer $n \geq 1$.

   You may use induction or any other method.

$$\sum_{i=1}^{2n} i - \sum_{i=1}^{n} i \;=\; \frac{n(3n+1)}{2}$$

2. For each one of the following three closed-form expressions, define a recursive formula with an initial value such that the solution to the recursive formula is the closed-form expression. That is, for each expression, define $T(n)$ as a function of $T(n-1)$ and define $T(1)$.

   Justify your answers.

   (a) $T(n) = 3n$ for an integer $n \geq 1$.

   (b) $T(n) = 3^n$ for an integer $n \geq 1$.

   (c) $T(n) = 3n!$ for an integer $n \geq 1$.

3. A legal password of length 4 must start and end with one of the five letters A, B, C, D, E while the middle two positions must be filled with one of the ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

Justify your answers to the following nine questions.

(a) How many legal passwords of length 4 are there?

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(b) How many legal passwords of length 4 are there in which the two letters must be the same and the two digits must be the same?

(c) How many legal passwords of length 4 are there in which the two letters must be the same while the two digits must be different?

(d) How many legal passwords of length 4 are there in which the two letters must be different while the two digits must be the same?

(e) How many legal passwords of length 4 are there in which the two letters must be different and the two digits must be different?

(f) How many legal passwords of length 4 are there in which the two letters may be the same or may be different while the two digits must be the same?

(g) How many legal passwords of length 4 are there in which the two letters may be the same or may be different while the two digits must be different?

(h) How many legal passwords of length 4 are there in which the two letters must be the same while the two digits may be the same or may be different?

(i) How many legal passwords of length 4 are there in which the two letters must be different while the two digits may be the same or may be different?

4. Prove the following identity for any two integers k and n such that $1 \leq k \leq n$:

$$\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$$

5. You have two 4-sided fair dice. The four faces of the first dice are labeled with the numbers $2, 4, 6, 8$ while the four faces of the second dice are labeled with the numbers $1, 3, 5, 7$. You throw both dice together.

   Justify your answers to the following questions.

   (a) What is the probability that the sum of the two shown numbers is even?

   (b) What is the probability that the product of the two shown numbers is even?

   (c) What is the probability that the sum of the two shown numbers is 7?

   (d) What is the probability that the product of the two shown numbers is 6?

(e) Given that the product of the two shown numbers is greater than 11, what is the probability that their sum is exactly 9?

(f) Given that the sum of the two shown numbers is less than 10, what is the probability that their product is less than 7?