CISC 2210 TR11 – Introduction to Discrete Structures

Midterm 2 Exam – Solutions

Nov 7, 2023

1. Prove the correctness of the following identity for any integer $n \geq 1$.

$$\sum_{i=1}^{2n} i - \sum_{i=1}^{n} i = \frac{n(3n+1)}{2}$$

Core identity:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

A direct proof I. By replacing the upper limit n with $2n$, the core identity becomes

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2}$$

The proof follows by the core identity and the above identity.

$$\begin{aligned}
\sum_{i=1}^{2n} i - \sum_{i=1}^{n} i &= \frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} \\
&= \frac{2n(2n+1) - n(n+1)}{2} \\
&= \frac{4n^2 + 2n - n^2 - n}{2} \\
&= \frac{3n^2 + n}{2} \\
&= \frac{n(3n+1)}{2}
\end{aligned}$$

A direct proof II. Evaluate the left side of the identity as follows using the core identity.

$$\begin{aligned}
\sum_{i=1}^{2n} i - \sum_{i=1}^{n} i &= (1 + 2 + \cdots + n + (n+1) + (n+2) + \cdots + 2n) - (1 + 2 + \cdots + n) \\
&= (n+1) + (n+2) + \cdots + 2n \\
&= (n+1) + (n+2) + \cdots + (n+n) \\
&= \sum_{i=1}^{n} (n+i) \\
&= \sum_{i=1}^{n} n + \sum_{i=1}^{n} i \\
&= n^2 + \frac{n(n+1)}{2} \\
&= \frac{2n^2 + n^2 + n}{2} \\
&= \frac{3n^2 + n}{2} \\
&= \frac{n(3n+1)}{2}
\end{aligned}$$

Proof by induction:

- *Notations.*

$$
\begin{aligned}
L(n) &= \sum_{i=1}^{2n} i - \sum_{i=1}^{n} i \\
R(n) &= \frac{n(3n+1)}{2}
\end{aligned}
$$

- *Induction base.* Prove that $L(1) = R(1)$:

$$L(1) = (1+2) - 1 = 2 = \frac{4}{2} = \frac{1 \cdot (3 \cdot 1 + 1)}{2} = R(1)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 1$:

$$\sum_{i=1}^{2k} i - \sum_{i=1}^{k} i = \frac{k(3k+1)}{2}$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 1$:

$$
\begin{aligned}
L(k+1) &= \sum_{i=1}^{2(k+1)} i - \sum_{i=1}^{k+1} i \\
&= \left(\sum_{i=1}^{2k} i + (2k+1) + (2k+2) \right) - \left(\sum_{i=1}^{k} i + (k+1) \right) \\
&= \sum_{i=1}^{2k} i - \sum_{i=1}^{k} i + ((2k+1) + (2k+2) - (k+1)) \\
&= L(k) + (3k+2) \\
&= R(k) + (3k+2) \\
&= \frac{k(3k+1)}{2} + (3k+2) \\
&= \frac{k(3k+1)}{2} + \frac{2(3k+2)}{2} \\
&= \frac{3k^2 + k + 6k + 4}{2} \\
&= \frac{3k^2 + 7k + 4}{2} \\
&= \frac{(k+1)(3k+4)}{2} \\
&= \frac{(k+1)(3(k+1)+1)}{2} \\
&= R(k+1)
\end{aligned}
$$

2. For each one of the following three closed-form expressions, define a recursive formula with an initial value such that the solution to the recursive formula is the closed-form expression. That is, for each expression, define $T(n)$ as a function of $T(n-1)$ and define $T(1)$.

   (a) $T(n) = 3n$ for an integer $n \geq 1$.

   Answer:

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ T(n-1) + 3 & \text{for } n > 1 \end{cases}$$

   Proof by induction $T(n) = 3n$ for $n \geq 1$:

   - *Induction base.* $T(1) = 3 \cdot 1 = 3$ for $n = 1$.
   - *Induction hypothesis.* Assume that $T(n-1) = 3(n-1)$ for $n > 1$.
   - *Inductive step.* Prove that $T(n) = 3n$ for $n > 1$:

$$\begin{aligned} T(n) &= T(n-1) + 3 \\ &= 3(n-1) + 3 \\ &= 3n - 3 + 3 \\ &= 3n \end{aligned}$$

   (b) $T(n) = 3^n$ for an integer $n \geq 1$.

   Answer:

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ 3T(n-1) & \text{for } n > 1 \end{cases}$$

   Proof by induction that $T(n) = 3^n$ for $n \geq 1$:

   - *Induction base.* $T(1) = 3^1 = 3$ for $n = 1$.
   - *Induction hypothesis.* Assume that $T(n-1) = 3^{n-1}$ for $n > 1$.
   - *Inductive step.* Prove that $T(n) = 3^n$ for $n > 1$:

$$\begin{aligned} T(n) &= 3T(n-1) \\ &= 3 \cdot 3^{n-1} \\ &= 3^n \end{aligned}$$

   (c) $T(n) = 3n!$ for an integer $n \geq 1$.

   Answer:

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ nT(n-1) & \text{for } n > 1 \end{cases}$$

   Proof by induction that $T(n) = 3n!$ for $n \geq 1$:

   - *Induction base.* $T(1) = 3 \cdot 1! = 3$ for $n = 1$.
   - *Induction hypothesis.* Assume that $T(n-1) = 3(n-1)!$ for $n > 1$.
   - *Inductive step.* Prove that $T(n) = 3n!$ for $n > 1$:

$$\begin{aligned} T(n) &= nT(n-1) \\ &= n \cdot 3(n-1)! \\ &= 3 \cdot n(n-1)! \\ &= 3n! \end{aligned}$$

3. A legal password of length 4 must start and end with one of the five letters A, B, C, D, E while the middle two positions must be filled with one of the ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

(a) How many legal passwords of length 4 are there?

Answer: $\mathbf{2500 = 5 \times 10 \times 10 \times 5}$

Proof: Since letters and digits may repeat, it follows that there are 5 options for the first position, 10 options for the second position, 10 options for the third position, and 5 options for the last position. In total there are $5 \times 10 \times 10 \times 5 = 2500$ different legal passwords of length 4.

(b) How many legal passwords of length 4 are there in which the two letters must be the same and the two digits must be the same?

Answer: $\mathbf{50 = 5 \times 10 \times 1 \times 1}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits must be the same, it follows that there is only 1 option for the third position. Since the two letters must be the same, it follows that there is only 1 option for the last position. In total there are $5 \times 10 \times 1 \times 1 = 50$ different legal passwords of length 4 with these additional constraints.

(c) How many legal passwords of length 4 are there in which the two letters must be the same while the two digits must be different?

Answer: $\mathbf{450 = 5 \times 10 \times 9 \times 1}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits must be different, it follows that there are only 9 options for the third position. Since the two letters must be the same, it follows that there is only 1 option for the last position. In total there are $5 \times 10 \times 9 \times 1 = 450$ different legal passwords of length 4 with these additional constraints.

(d) How many legal passwords of length 4 are there in which the two letters must be different while the two digits must be the same?

Answer: $\mathbf{200 = 5 \times 10 \times 1 \times 4}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits must be the same, it follows that there is only 1 option for the third position. Since the two letters must be different, it follows that there are only 4 options for the last position. In total there are $5 \times 10 \times 1 \times 4 = 200$ different legal passwords of length 4 with these additional constraints.

(e) How many legal passwords of length 4 are there in which the two letters must be different and the two digits must be different?

Answer: $\mathbf{1800 = 5 \times 10 \times 9 \times 4}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits must be different, it follows that there are only 9 options for the third position. Since the two letters must be different, it follows that there are only 4 options for the last position. In total there are $5 \times 10 \times 9 \times 4 = 1800$ different legal passwords of length 4 with these additional constraints.

(f) How many legal passwords of length 4 are there in which the two letters may be the same or may be different while the two digits must be the same?

Answer: $\mathbf{250 = 5 \times 10 \times 1 \times 5}$

Proof: Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits must be the same, it follows that there is only 1 option for the third position. Since the two letters may be the same or may be different, it follows that there are 5 options for the last position. In total there are $5 \times 10 \times 1 \times 5 = 250$ different legal passwords of length 4 with these additional constraints.

(g) How many legal passwords of length 4 are there in which the two letters may be the same or may be different while the two digits must be different?

Answer: $\mathbf{2250 = 5 \times 10 \times 9 \times 5}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits must be different, it follows that there are only 9 options for the third position. Since the two letters may be the same or may be different, it follows that there are 5 options for the last position. In total there are $5 \times 10 \times 9 \times 5 = 2250$ different legal passwords of length 4 with these additional constraints.

(h) How many legal passwords of length 4 are there in which the two letters must be the same while the two digits may be the same or may be different?

Answer: $\mathbf{500 = 5 \times 10 \times 10 \times 1}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits may be the same or may be different, it follows that there are 10 options for the third position. Since the two letters must be the same, it follows that there is only 1 option for the last position. In total there are $5 \times 10 \times 10 \times 1 = 500$ different legal passwords of length 4 with these additional constraints.

(i) How many legal passwords of length 4 are there in which the two letters must be different while the two digits may be the same or may be different?

Answer: $\mathbf{2000 = 5 \times 10 \times 10 \times 4}$

Proof: There are 5 options for the first position and 10 options for the second position. Since the two digits may be the same or may be different, it follows that there are 10 options for the third position. Since the two letters must be different, it follows that there are only 4 options for the last position. In total there are $5 \times 10 \times 10 \times 4 = 2000$ different legal passwords of length 4 with these additional constraints.

Remarks: Denote by $a, b, c, d, e, f, g, h, i$ the answers to parts (a), (b), (c), (d), (e), (f), (g), (h), and (i) respectively. The following are some relationships among the nine answers that can provide alternative proofs to some of the answers.

(1) $a = b + c + d + e$. Indeed $2500 = 50 + 450 + 200 + 1800$.
(2) $a = f + g$. Indeed $2500 = 250 + 2250$.
(3) $a = h + i$. Indeed $2500 = 500 + 2000$.
(4) $f = b + d$. Indeed $250 = 50 + 200$.
(5) $g = c + e$. Indeed $2250 = 450 + 1800$.
(6) $h = b + c$. Indeed $500 = 50 + 450$.
(7) $i = d + e$. Indeed $2000 = 200 + 1800$.

In particular, after computing the answers to (a), (f), and (h), the other six answers can be found by identities (2) through (7) as follows. First, identity (2) implies the answer to (g). Next, identity (3) implies the answer to (i). Finally, the last four identities represent four equations with four variables whose values are the answers to parts (b), (c), (d), and (e). Note, that identity (1) is not needed since it can be deduced from the other six identities.

4. Prove the following identity for any two integers k and n such that $1 \leq k \leq n$:

$$\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$$

Proof: By definition,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It follows that,

$$\begin{aligned}\binom{n+1}{k+1} &= \frac{(n+1)!}{(k+1)!((n+1)-(k+1))!} \\ &= \frac{(n+1)!}{(k+1)!(n-k)!}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{n+1}{k+1}\binom{n}{k} &= \frac{n+1}{k+1} \cdot \frac{n!}{k!(n-k)!} \\ &= \frac{(n+1)n!}{(k+1)k!(n-k)!} \\ &= \frac{(n+1)!}{(k+1)!(n-k)!} \\ &= \binom{n+1}{k+1}\end{aligned}$$

5. You have two 4-sided fair dice. The four faces of the first dice are labeled with the numbers $2, 4, 6, 8$ while the four faces of the second dice are labeled with the numbers $1, 3, 5, 7$. You throw both dice together.

Preprocessing: The tables below list the 16 possible sums and products of the two shown numbers.

Sum	2	4	6	8
1	3	5	**7**	9
3	5	**7**	9	11
5	**7**	9	11	13
7	9	11	13	15

Product	2	4	6	8
1	2	4	**6**	8
3	**6**	12	18	24
5	10	20	30	40
7	14	28	42	56

(a) What is the probability that the sum of the two shown numbers is even?

Answer: $\mathbf{\frac{0}{16} = 0}$ **(0%)**

Proof: The sum of an even number and an odd number is always odd. Therefore, the sum of the two shown numbers is even with probability 0. Indeed, in the above sum-table none of the 16 sums is even implying that the probability of an even sum is $0/16 = 0$.

(b) What is the probability that the product of the two shown numbers is even?

Answer: $\mathbf{\frac{16}{16} = 1}$ **(100%)**

Proof: The product of an even number and an odd number is always even. Therefore, the product of the two shown numbers is even with probability 1. Indeed, in the above product-table all the 16 products are even implying that the probability of an even product is $16/16 = 1$.

(c) What is the probability that the sum of the two shown numbers is 7?

Answer: $\mathbf{\frac{3}{16} = 0.1875}$ **(18.75%)**

Proof: Out of the 16 possible pairs of shown numbers, only in the following 3 the sum of the two shown numbers is 7:

$$(1, 6) \quad (3, 4) \quad (5, 2)$$

Hence, the probability that the sum of the two shown numbers is 7 is $3/16$.

(d) What is the probability that the product of the two shown numbers is 6?

Answer: $\mathbf{\frac{2}{16} = \frac{1}{8} = 0.125}$ **(12.5%)**

Proof: Out of the 16 possible pairs of shown numbers, only in the following 2 the product of the two shown numbers is 6:

$$(1, 6) \quad (3, 2)$$

Hence, the probability that the product of the two shown numbers is 6 is $2/16 = 1/8$.

(e) Given that the product of the two shown numbers is greater than 11, what is the probability that their sum is exactly 9?

Answer: $\mathbf{\frac{3}{10} = 0.3}$ **(30%)**

Proof: Out of the 16 possible pairs of shown numbers, only in the following 10 pairs the product of the two numbers is greater than 11:

$$(3,4) \quad (3,6) \quad (3,8) \quad (5,4) \quad (5,6) \quad (5,8) \quad (7,2) \quad (7,4) \quad (7,6) \quad (7,8)$$

Out of these 10 pairs, only in the following 3 pairs the sum of the two numbers is exactly 9:

$$(3,6) \quad (5,4) \quad (7,2)$$

Hence, the probability that the sum of the two shown numbers is exactly 9 given that their product is greater than 11 is $3/10$.

(f) Given that the sum of the two shown numbers is less than 10, what is the probability that their product is less than 7?

Answer: $\mathbf{\frac{4}{10} = 0.4}$ **(40%)**

Proof: Out of the 16 possible pairs of shown numbers, only in the following 10 pairs the sum of the two numbers is less than 10:

$$(1,2) \quad (1,4) \quad (1,6) \quad (1,8) \quad (3,2) \quad (3,4) \quad (3,6) \quad (5,2) \quad (5,4) \quad (7,2)$$

Out of these 10 pairs, only in the following 4 pairs the product of the two numbers is less than 7:

$$(1,2) \quad (1,4) \quad (1,6) \quad (3,2)$$

Hence, the probability that the product of the two shown numbers is less than 7 given that their sum is less than 10 is $4/10 = 2/5$.

Remark: The answers can be found using probability arguments. In particular, Bayes' theorem for conditional probabilities can be applied to find the answers to parts (e) and (f). However, with this particular problem, it is easier to find counting arguments to solve the six problems. Mainly because the sample space which has only 16 outcomes is relatively small.