CISC 2210 (TR11) – Introduction to Discrete Structures

Midterm 1 Exam – Solutions

September 25, 2025

1. Let A, B, and C be three sets whose members are taken from a universal set $\mathcal{U}$.

Question I: Assume $(\neg A) \cap (\neg B) \cap (\neg C) = \emptyset$. What is $A \cup B \cup C$?

Answer I: $A \cup B \cup C = \mathcal{U}$.

Question II: Assume $(\neg A) \cup (\neg B) \cup (\neg C) = \emptyset$. What can you say about the sets A, B, and C?

Answer II: $A = B = C = \mathcal{U}$.

Direct proofs:

First proof of answer I: If an intersection of sets is the empty set, then it must be the case that every member of the universal set does not belong to at least one set in the intersection and therefore belongs to the complement of that particular set. As a result, the union of the complement sets of all the sets in the intersection must be the universal set.

Since $\neg(\neg(S)) = S$ for any set S, applying the above argument on the assumption $(\neg A) \cap (\neg B) \cap (\neg C) = \emptyset$ implies that

$$A \cup B \cup C = \mathcal{U}$$

First proof of answer II: If a union of sets is the empty set, then it must be the case that each set in the union is also the empty set. By assumption, $(\neg A) \cap (\neg B) \cap (\neg C) = \emptyset$. Therefore,

$$(\neg A) = (\neg B) = (\neg C) = \emptyset$$

Since $\neg(\neg(S)) = S$ for any set S, complementing all the terms in the above identities completes the proof,

$$A = B = C = \mathcal{U}$$

Proofs using the De Morgan's laws:

Second proof of answer I: One of the De Morgan's laws states that the intersection of the complements of three sets is the complement of the union of these three sets:

$$(\neg A) \cap (\neg B) \cap (\neg C) = \neg(A \cup B \cup C)$$

Since by assumption $(\neg A) \cap (\neg B) \cap (\neg C) = \emptyset$, the above identity implies that also

$$\neg(A \cup B \cup C) = \emptyset$$

The following identity is implied by complementing both sides of the above identity:

$$A \cup B \cup C = \neg(\emptyset) = \mathcal{U}$$

Second proof of answer II: One of the De Morgan's laws states that the union of the complements of three sets is the complement of the intersection of these three sets:

$$(\neg A) \cup (\neg B) \cup (\neg C) = \neg(A \cap B \cap C)$$

Since by assumption $(\neg A) \cup (\neg B) \cup (\neg C) = \emptyset$, the above identity implies that also

$$\neg(A \cap B \cap C) = \emptyset$$

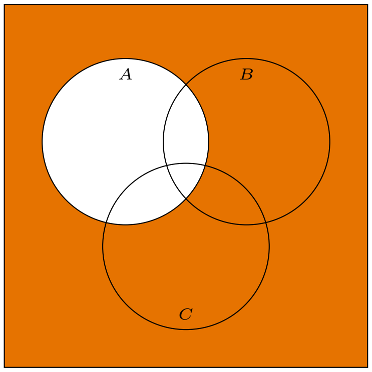
The following identity is implied by complementing both sides of the above identity:

$$A \cap B \cap C = (\neg\emptyset) = \mathcal{U}$$

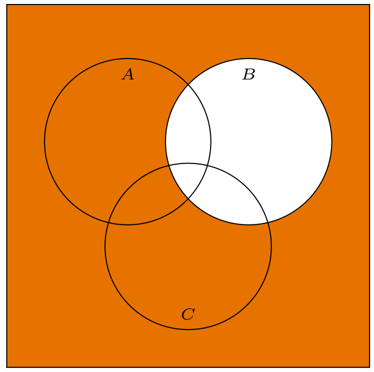
But if the intersection of sets is the universal set, it must be the case that each set in the intersection is the universal set. In conclusion,

$$A = B = C = \mathcal{U}$$

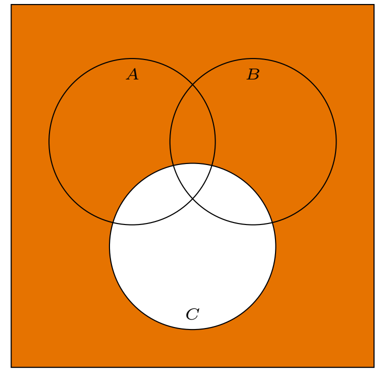
Proofs using Venn diagrams:



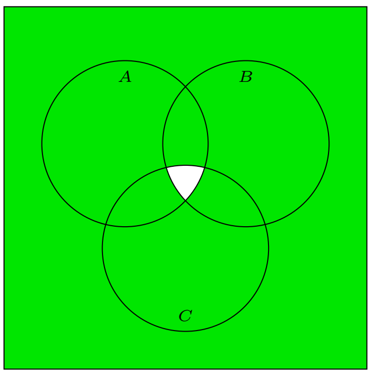
$\neg A$



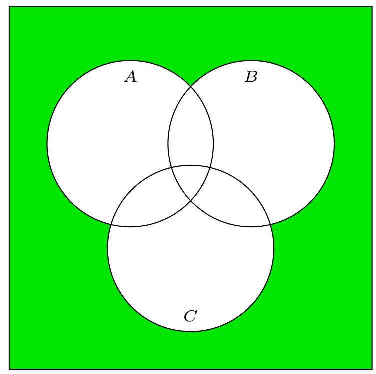
$\neg B$



$\neg C$



$(\neg A) \cup (\neg B) \cup (\neg C)$

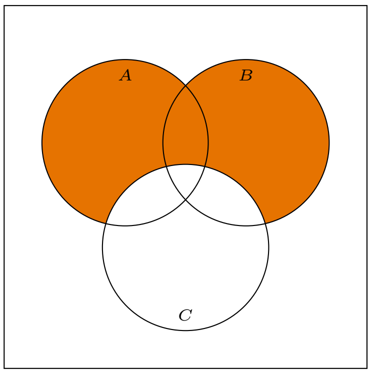


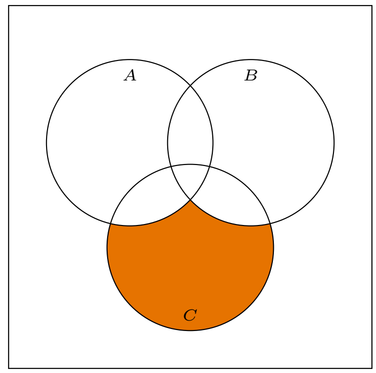
$(\neg A) \cap (\neg B) \cap (\neg C)$

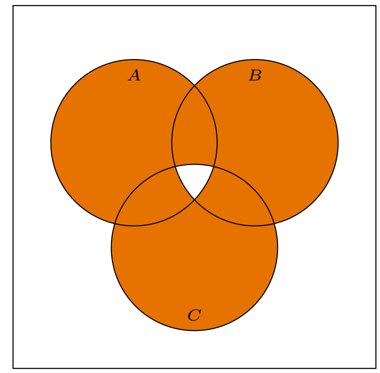
Third proof of answer I: By assumption, the shaded area in the right diagram of the above figure is empty. As a result, the non-shaded area, which is $A \cup B \cup C$, must be the universal set $\mathcal{U}$.

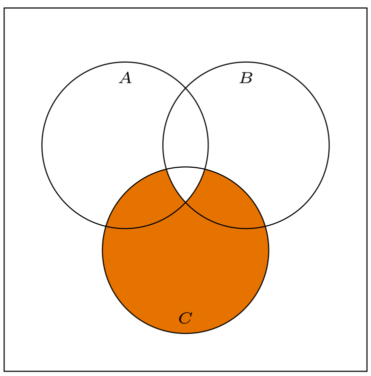
Third proof of answer II: By assumption, the shaded area in the left diagram of the above figure is empty. As a result, the non-shaded area, which is $A \cap B \cap C$, must be the universal set $\mathcal{U}$. But if the intersection of sets is the universal set, it must be the case that each set in the intersection is the universal set.

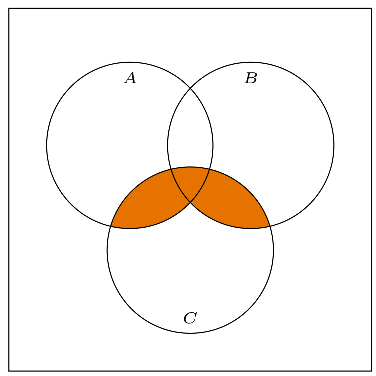
2. Match the 4 expressions on the next page with their corresponding Venn diagrams from the 12 provided below.

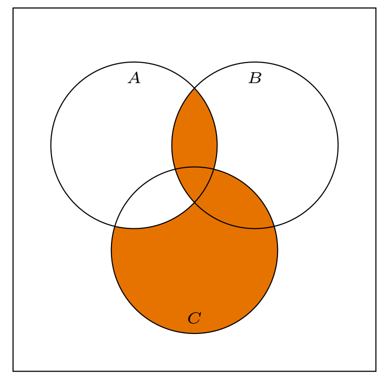


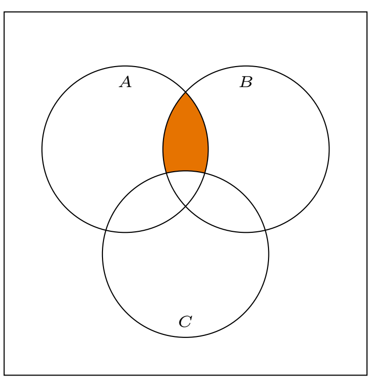


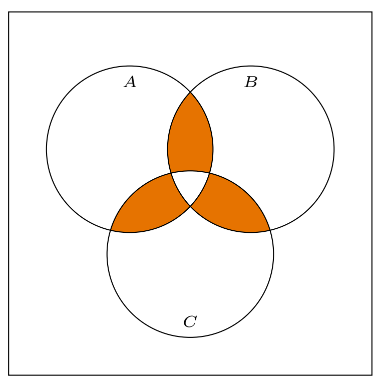


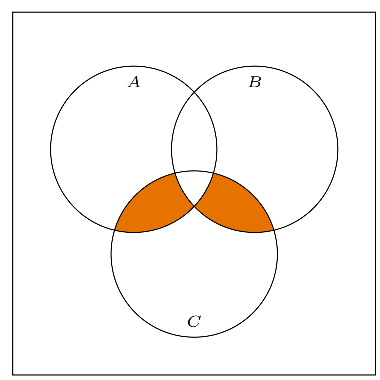


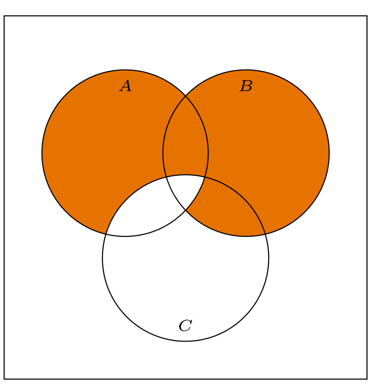


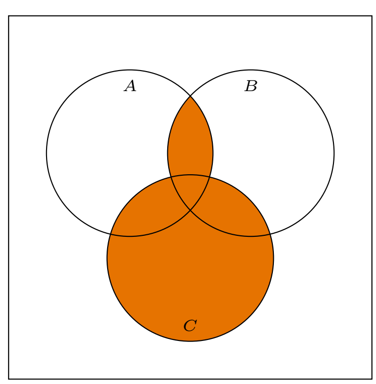


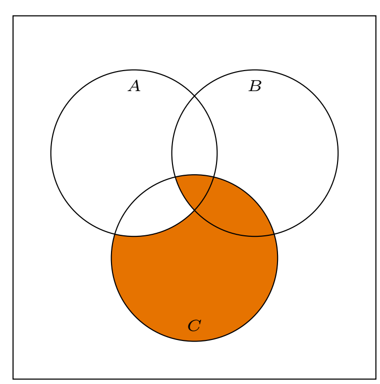




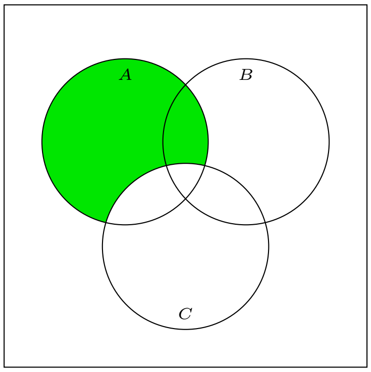




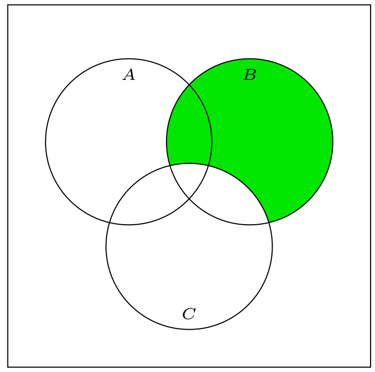




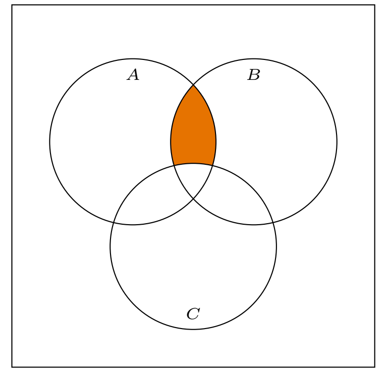
(a) The Venn Diagram in **Row 3 Column 1** represents the expression: $(\mathbf{A} \setminus \mathbf{C}) \cap (\mathbf{B} \setminus \mathbf{C})$.



$A \setminus C$

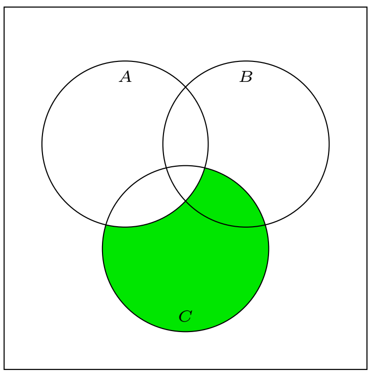


$B \setminus C$

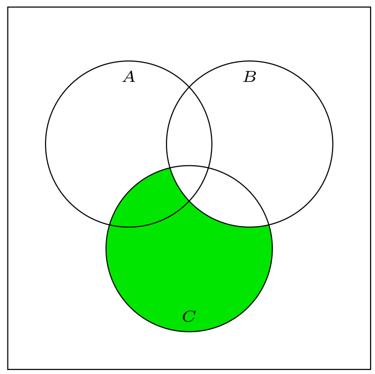


$(A \setminus C) \cap (B \setminus C)$

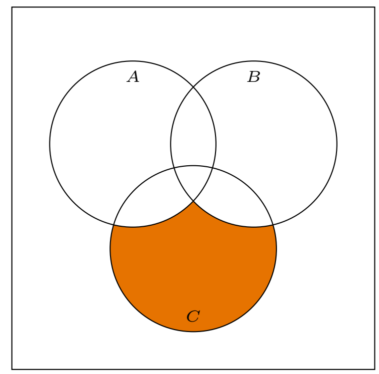
(b) The Venn Diagram in **Row 1 Column 2** represents the expression: $(\mathbf{C} \setminus \mathbf{A}) \cap (\mathbf{C} \setminus \mathbf{B})$.



$C \setminus A$

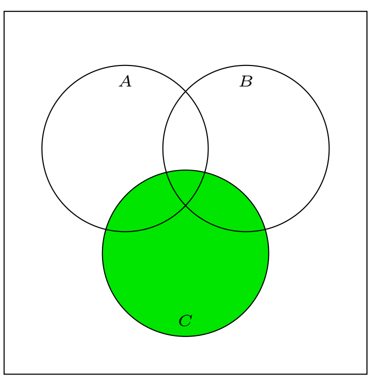


$C \setminus B$

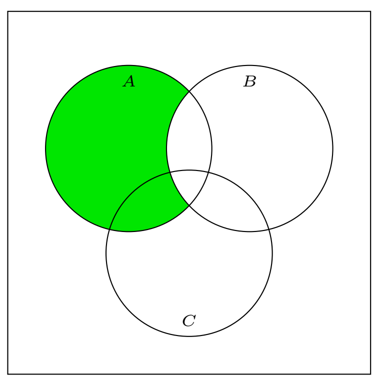


$(C \setminus A) \cap (C \setminus B)$

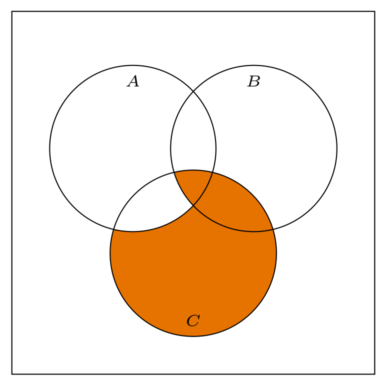
(c) The Venn Diagram in **Row 4 Column 3** represents the expression: $\mathbf{C} \setminus (\mathbf{A} \setminus \mathbf{B})$.



C

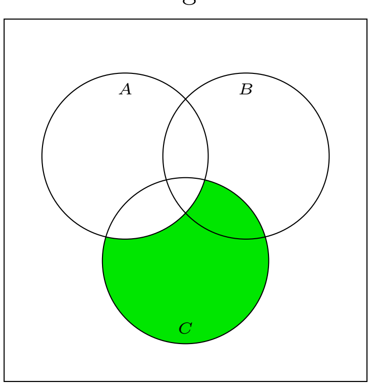


$A \setminus B$

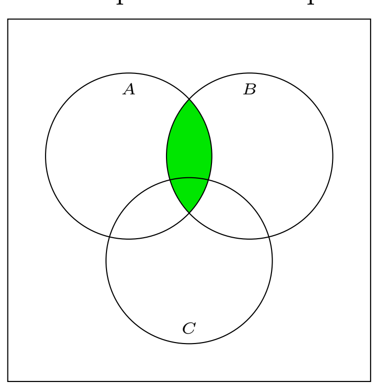


$C \setminus (A \setminus B)$

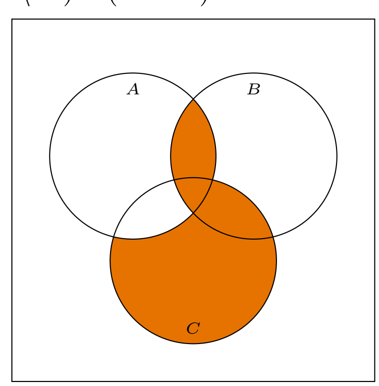
(d) The Venn Diagram in **Row 2 Column 3** represents the expression: $(\mathbf{C} \setminus \mathbf{A}) \cup (\mathbf{A} \cap \mathbf{B})$.



$C \setminus A$



$A \cap B$



$(C \setminus A) \cup (A \cap B)$

3. At a cat show, every cat wore at least one ribbon. The available colors were Blue, Red, and Green. Based on the data below, determine the total number of cats at the show.
   - Every cat wore ribbons of at least two different colors.
   - 80 cats wore both Blue and Red ribbons (and maybe also a Green ribbon).
   - 60 cats wore both Red and Green ribbons (and maybe also a Blue ribbon).
   - 40 cats wore both Green and Blue ribbon (and maybe also a Red ribbon).
   - 20 cats wore ribbons of all three colors.

Answer: 140.

Notations: Let B be the set of cats who wore a Blue ribbon, let R be the set of cats who wore a Red ribbon, and let G be the set of cats who wore a Green ribbon.

Proof using a Venn diagram: The Venn diagram below reflects the data above.

- Since each cat wore ribbons of at least two different colors, it follows that

$$|B \cap \overline{R} \cap \overline{G}| = |\overline{B} \cap R \cap \overline{G}| = |\overline{B} \cap \overline{R} \cap G| = 0$$

- Since 80 cats wore both Blue and Red ribbons while 20 cats wore ribbons of all three colors, it follows that 60 cats wore a Blue ribbon and a Red ribbon but did not wear a Green ribbon:

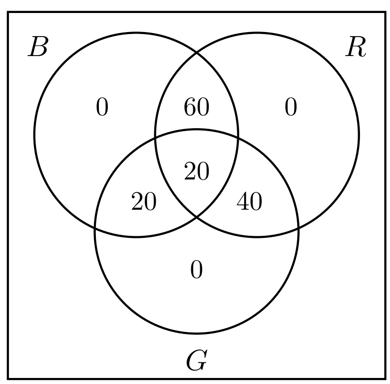
$$|B \cap R \cap \overline{G}| = 80 - 20 = 60$$

- Since 60 cats wore both Red and Green ribbons while 20 cats wore ribbons of all three colors, it follows that 40 cats wore a Red ribbon and a Green ribbon but did not wear a Blue ribbon:

$$|\overline{B} \cap R \cap G| = 60 - 20 = 40$$

- Since 40 cats wore both Green and Blue ribbons while 20 cats wore ribbons of all three colors, it follows that 20 cats wore a Green ribbon and a Blue ribbon but did not wear a Red ribbon:

$$|B \cap \overline{R} \cap G| = 40 - 20 = 20$$



As a result, the total number of cats at the show is $140 = 60 + 40 + 20 + 20$.

Proof using the Principle of Inclusion Exclusion:

- Applying the principle of inclusion exclusion on the set B implies that

$$|B| = |B \cap R| + |B \cap G| - |B \cap R \cap G| = 80 + 40 - 20 = 100$$

- Applying the principle of inclusion exclusion on the set R implies that

$$|R| = |R \cap B| + |R \cap G| - |B \cap R \cap G| = 80 + 60 - 20 = 120$$

- Applying the principle of inclusion exclusion on the set G implies that

$$|G| = |G \cap B| + |G \cap R| - |B \cap R \cap G| = 40 + 60 - 20 = 80$$

- Applying the principle of inclusion exclusion on the set $B \cup R \cup G$ implies that

$$\begin{aligned} |B \cup R \cup G| &= |B| + |R| + |G| - |B \cap R| - |R \cap G| - |G \cap B| + |B \cup G \cup R| \\ &= 100 + 120 + 80 - 80 - 60 - 40 + 20 = 140 \end{aligned}$$

At a dog show, every dog wore at least one ribbon. The available colors were Blue, Red, and Green. Based on the data below, determine how many dogs wore ribbons of all three colors.

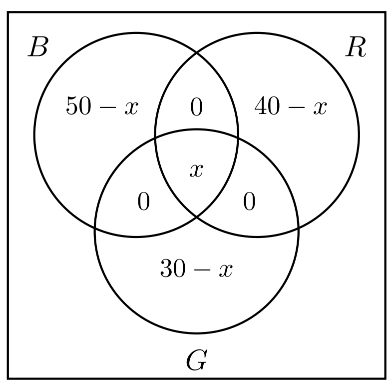
- There were 100 dogs in total.
- No dog wore ribbons of exactly two different colors.
- 50 dogs wore a Blue ribbon (and maybe also a Red ribbon and a Green ribbon).
- 40 dogs wore a Red ribbon (and maybe also a Blue ribbon and a Green ribbon).
- 30 dogs wore a Green ribbon (and maybe also a Blue ribbon and a Red ribbon).

Answer: **10**.

Notations: Let B be the set of dogs who wore a Blue ribbon, let R be the set of dogs who wore a Red ribbon, and let G be the set of dogs who wore a Green ribbon. Let $x = |B \cap R \cap G|$ denote the number of dogs who wore ribbons of all three colors.

Proof using a Venn diagram: The Venn diagram below reflects the data above.

- Since no dog wore ribbons of exactly two different colors, it follows that
$$|B \cap R \cap \overline{G}| = |B \cap \overline{R} \cap G| = |\overline{B} \cap R \cap G| = 0$$
- Since 50 dogs wore a Blue ribbon while x dogs wore ribbons of all three colors, it follows that $50 - x$ dogs wore only a Blue ribbon:
$$|B \cap \overline{R} \cap \overline{G}| = 50 - x$$
- Since 40 dogs wore a Red ribbon while x dogs wore ribbons of all three colors, it follows that $40 - x$ dogs wore only a Red ribbon:
$$|\overline{B} \cap R \cap \overline{G}| = 40 - x$$
- Since 30 dogs wore a Green ribbon while x dogs wore ribbons of all three colors, it follows that $30 - x$ dogs wore only a Green ribbon:
$$|\overline{B} \cap \overline{R} \cap G| = 30 - x$$



Since the dog show had in total 100 dogs, it follows that
$$100 = (50 - x) + (40 - x) + (30 - x) + x = 120 - 2x$$
The above identity is equivalent to $x = (120 - 100)/2 = 10$.

Proof using the Principle of Inclusion Exclusion: Since no dog wore ribbons of exactly two different colors, it follows that
$$|B \cap R| = |B \cap G| = |R \cap G| = |B \cap R \cap G| = x$$
The principle of inclusion exclusion for the set $B \cup R \cup G$ is
$$|B \cup R \cup G| = |B| + |R| + |G| - |B \cap R| - |B \cap G| - |R \cap G| + |B \cap R \cap G|$$
Plugging the data and the cumputed numbers for the sizes of the sets in the above identity implies that
$$100 = 50 + 40 + 30 - x - x - x + x = 120 - 2x$$
The above identity is equivalent to $x = (120 - 100)/2 = 10$.

4. **Proposition 1:** $L = x \vee \neg(y \vee z)$ is not equivalent to $R = x \wedge \neg(y \wedge z)$.

Proof: Either of the following two counterexamples is sufficient to prove that L and R are not equivalent.

(a) Consider the assignment $x = y = z = T$. It follows that $y \wedge z$ is True, which implies that $\neg(y \wedge z)$ is False, which implies that $R = x \wedge \neg(y \wedge z)$ is False. But since $x = T$, it follows that L is True. Due to this assignment, L is not equivalent to R.

(b) Consider the assignment $x = y = z = F$. It follows that $y \vee z$ is False, which implies that $\neg(y \vee z)$ is True, which implies that $L = x \vee \neg(y \vee z)$ is True. But since $x = F$, it follows that R is False. Due to this assignment, L is not equivalent to R.

Proof with truth tables: L and R are not equivalent because their respective truth tables (see below) have different final columns.

x	y	z	$y \vee z$	$\neg(y \vee z)$	$L = x \vee \neg(y \vee z)$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	T

x	y	z	$y \wedge z$	$\neg(y \wedge z)$	$R = x \wedge \neg(y \wedge z)$
T	T	T	T	F	F
T	T	F	F	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

Remark: The truth tables for L and R confirm that there are no counterexamples other than the two identified in the first proof.

Proposition 2: $L = (x \wedge \neg y) \vee (\neg x \wedge z)$ is equivalent to $R = (x \vee z) \wedge (\neg x \vee \neg y)$.

Proof: The proposition follows because $L \equiv R$ when $x = T$ and when $x = F$.

(a) Assigning $x = T$ and $\neg x = F$ in L and R imply that

$$\begin{aligned} L = (x \wedge \neg y) \vee (\neg x \wedge z) &= (T \wedge \neg y) \vee (F \wedge z) = \neg y \vee F = \neg y \\ R = (x \vee z) \wedge (\neg x \vee \neg y) &= (T \vee z) \wedge (F \vee \neg y) = T \wedge \neg y = \neg y \end{aligned}$$

(b) Assigning $x = F$ and $\neg x = T$ in L and R imply that

$$\begin{aligned} L = (x \wedge \neg y) \vee (\neg x \wedge z) &= (F \wedge \neg y) \vee (T \wedge z) = F \vee z = z \\ R = (x \vee z) \wedge (\neg x \vee \neg y) &= (F \vee z) \wedge (T \vee \neg y) = z \wedge T = z \end{aligned}$$

Proof with truth tables: $L \equiv R$ because their respective truth tables (see below) have identical final columns.

x	y	z	$x \wedge \neg y$	$\neg x \wedge z$	$(x \wedge \neg y) \vee (\neg x \wedge z)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	F	F

x	y	z	$x \vee z$	$\neg x \vee \neg y$	$(x \vee z) \wedge (\neg x \vee \neg y)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	F	T	F

5. The table below shows which sports are played by seven students. A T in the table means the student in that column plays the sport in that row. For example, the T in the (Basketball,Bob) entry means Bob plays Basketball while the F in the (Football,Carol) entry means Carol does not play Football.

Sport \ Student	Alice	Bob	Carol	David	Eva	Frank	Gina
Basketball	F	T	T	T	F	T	F
Football	T	T	F	F	T	F	F
Tennis	F	T	T	T	T	F	F

Let X be the set containing the 7 students and let Y be the set containing the 3 sports:

$$X = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{David}, \text{Eva}, \text{Frank}, \text{Gina}\} \quad Y = \{\text{Basketball}, \text{Football}, \text{Tennis}\}$$

(a) $\forall_{x \in X} \forall_{y \in Y}$ (x plays y) is **False** because there exists at least one F-entry in the table.
   - For example, Alice does not play basketball.

(b) $\exists_{x \in X} \exists_{y \in Y}$ (x plays y) is **True** because there exists at least one T-entry in the table.
   - For example, Carol plays Tennis.

(c) $\forall_{x \in X} \exists_{y \in Y}$ (x plays y) is **False** because there exists at least one all F-entry column in the table.
   - For example, Gina does not play any of the three sports.

(d) $\forall_{y \in Y} \exists_{x \in X}$ (x plays y) is **True** because there exists at least one T-entry in each row of the table.
   - For example, Carol plays basketball, Eva plays Football, and David plays Tennis.

(e) $\exists_{x \in X} \forall_{y \in Y}$ (x plays y) is **True** because there exists at least one all T-entry column in the table.
   - For example, Bob plays all three sports.

(f) $\exists_{y \in Y} \forall_{x \in X}$ (x plays y) is **False** because there exists at least one F-entry in each row of the table.
   - For example, Eva does not play Basketball, Carol does not play Football, and Frank does not play Tennis.

Remarks: Some answers imply other answers.

- If expression (a) is False, then expression (b) must be True. But if expression (b) is True, then expression (a) could be either True or False.
- If either expression (c) is False or expression (f) is False then expression (a) must be False. But if expression (a) is False, then both expression (c) and expression (f) could be either True or False.
- If expression (c) is False, then expression (f) must be False. But if expression (f) is False, then expression (c) could be either True or False.
- If expression (e) is True, then expression (d) must be True. But if expression (d) is True, then expression (e) could be either True or False.

6. You meet A, B, and C. One of them is a truth-teller who always tells the truth, one of them is a liar who always lies, and one of them is a mixer who sometimes tells the truth and sometimes lies.

   - A speaks first: "I am not a truth-teller".
   - B speaks second: "I am not a mixer".
   - C is silent.

   Who is the truth-teller? Who is the liar? And who is the mixer?

   Answer: B is the truth-teller, C is the liar, and A is the mixer.

   Explanation I: A cannot be the truth-teller, because that would make A's statement false – a contradiction. Nor can A be the liar, as that would make A's statement true – also a conradiction. The only remaining possibility is that A is the mixer. This makes B's statement true, meaning B is the truth-teller. Consequently, C must be the liar.

   Explanation II: A cannot be the truth-teller, because that would make A's statement false – a contradiction. Therfore, A's statement must be true, which implies that A is the mixer, not the liar. This makes B's statement true, meaning B is the truth-teller. Consequently, C must be the liar.

   Comparing both explanations: The above two explanations are nearly identical; they only differ in their reasoning for why A cannot be the liar

   Remark: A proof is incomplete if it only shows that one truth assignment works. For instance, B's statement being true while A's and C's are false. To be complete, the proof must also demonstrate that this truth assignment is the only one possible.