CISC 2210 (TR2) – Introduction to Discrete Structures

Midterm 1 Exam – Solutions

September 25, 2025

1. Let A, B, and C be three sets whose members are taken from a universal set $\mathcal{U}$.

Question I: Assume $(\neg A) \cup (\neg B) \cup (\neg C) = \mathcal{U}$. What is $A \cap B \cap C$?

Answer I: $A \cap B \cap C = \emptyset$

Question II: Assume $(\neg A) \cap (\neg B) \cap (\neg C) = \mathcal{U}$. What can you say about the sets A, B, and C?

Answer II: $A = B = C = \emptyset$

Direct proofs:

First proof of answer I: If a union of sets is the universal set, then it must be the case that every member of the universal set belongs to at least one set in the union and therefore does not belong to the complement of that particular set. As a result, the intersection of the complement sets of all the sets in the union must be empty.

Since $\neg(\neg(S)) = S$ for any set S, applying the above argument on the assumption $(\neg A) \cup (\neg B) \cup (\neg C) = \mathcal{U}$ implies that

$$A \cap B \cap C = \emptyset$$

First proof of answer II: If an intersection of sets is the universal set, then it must be the case that each set in the intersection is also the universal set. By assumption, $(\neg A) \cap (\neg B) \cap (\neg C) = \mathcal{U}$. Therefore,

$$(\neg A) = (\neg B) = (\neg C) = \mathcal{U}$$

Since $\neg(\neg(S)) = S$ for any set S, complementing all the terms in the above identities completes the proof,

$$A = B = C = \emptyset$$

Proofs using the De Morgan's laws:

Second proof of answer I: One of the De Morgan's laws states that the union of the complements of three sets is the complement of the intersection of these three sets:

$$(\neg A) \cup (\neg B) \cup (\neg C) = \neg(A \cap B \cap C)$$

Since by assumption $(\neg A) \cup (\neg B) \cup (\neg C) = \mathcal{U}$, the above identity implies that also

$$\neg(A \cap B \cap C) = \mathcal{U}$$

The following identity is implied by complementing both sides of the above identity:

$$A \cap B \cap C = (\neg \mathcal{U}) = \emptyset$$

Second proof of answer II: One of the De Morgan's laws states that the intersection of the complements of three sets is the complement of the union of these three sets:

$$(\neg A) \cap (\neg B) \cap (\neg C) = \neg(A \cup B \cup C)$$

Since by assumption $(\neg A) \cap (\neg B) \cap (\neg C) = \mathcal{U}$, the above identity implies that also

$$\neg(A \cup B \cup C) = \mathcal{U}$$

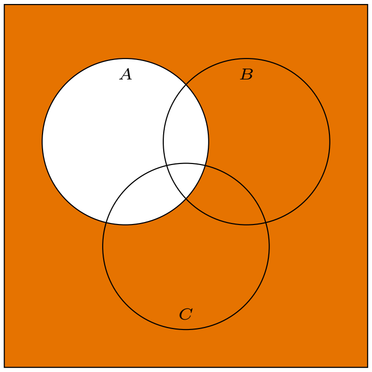
The following identity is implied by complementing both sides of the above identity:

$$A \cup B \cup C = \neg(\mathcal{U}) = \emptyset$$

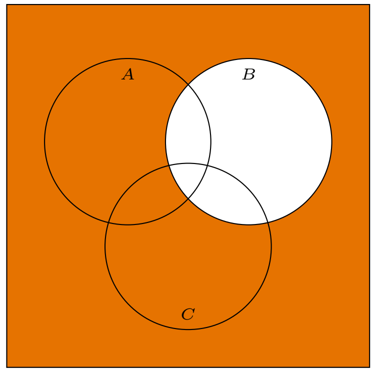
But if the union of sets is the empty set, it must be the case that each set in the union is the empty set. In conclusion,

$$A = B = C = \emptyset$$

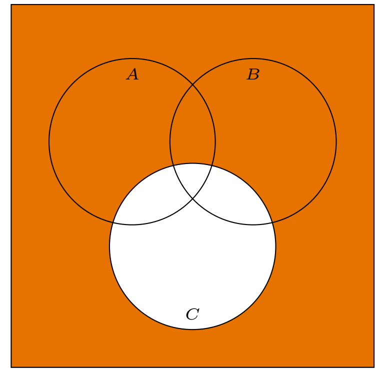
Proofs using Venn diagrams:



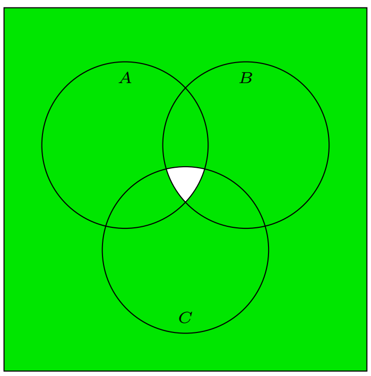
$\neg A$



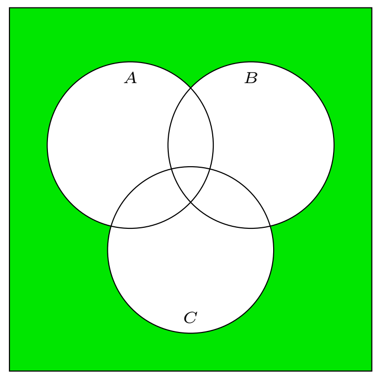
$\neg B$



$\neg C$



$(\neg A) \cup (\neg B) \cup (\neg C)$

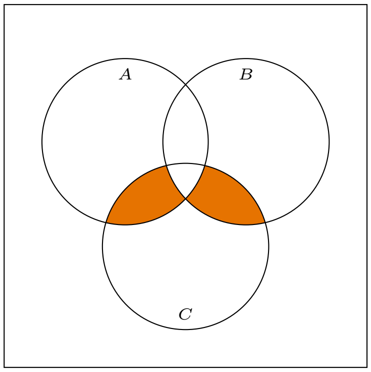


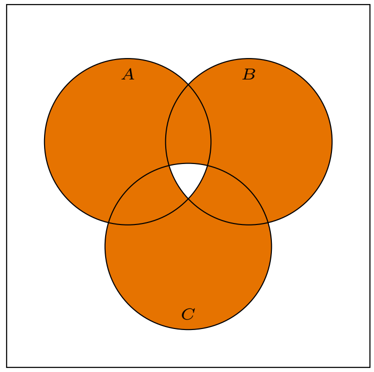
$(\neg A) \cap (\neg B) \cap (\neg C)$

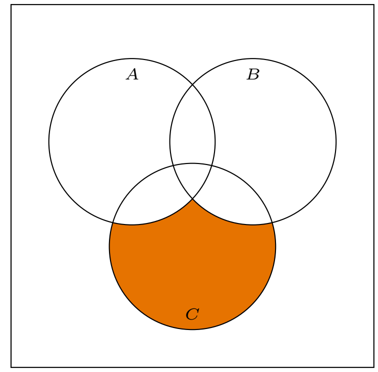
Third proof of answer I: By assumption, the shaded area in the left diagram of the above figure is $\mathcal{U}$. As a result, the non-shaded area, which is $A \cap B \cap C$, must be empty.

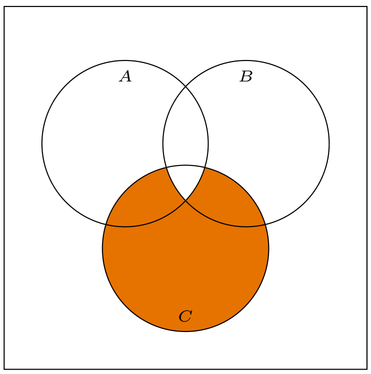
Third proof of answer II: By assumption, the shaded area in the right diagram of the above figure is $\mathcal{U}$. As a result, the non-shaded area, which is $A \cup B \cup C$, must be empty. But if the union of sets is the empty set, it must be the case that each set in the union is the empty set.

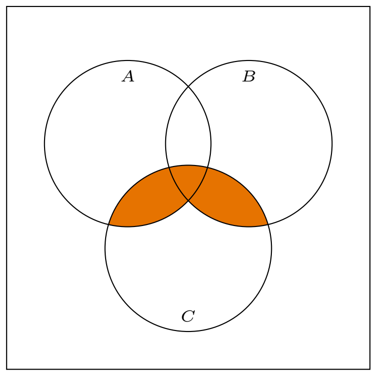
2. Match the 4 expressions on the next page with their corresponding Venn diagrams from the 12 provided below.

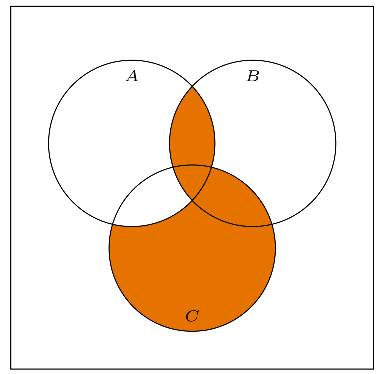


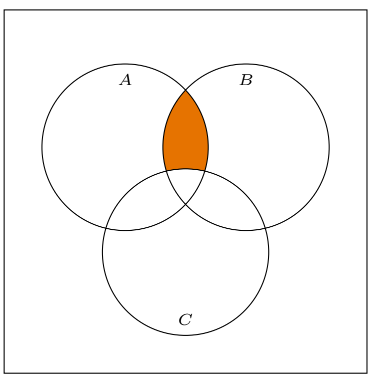


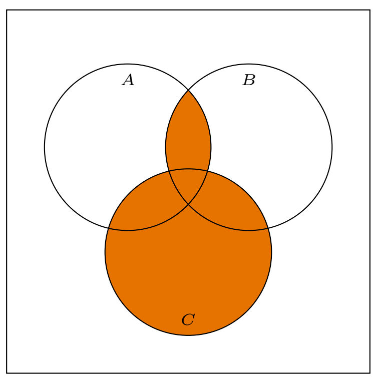


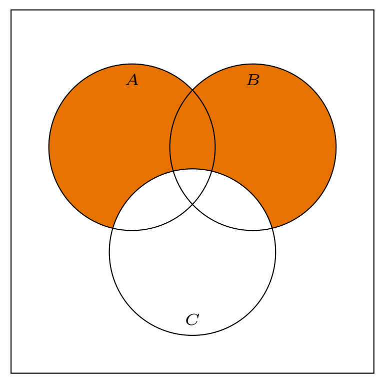


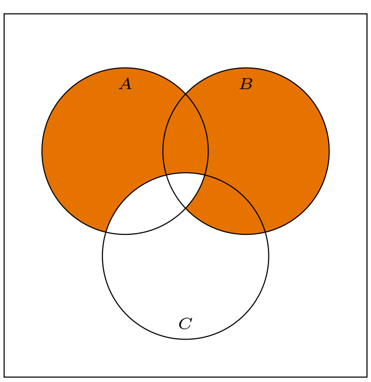


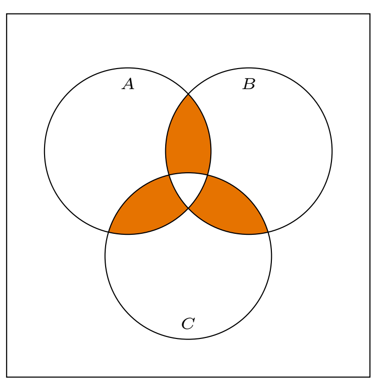


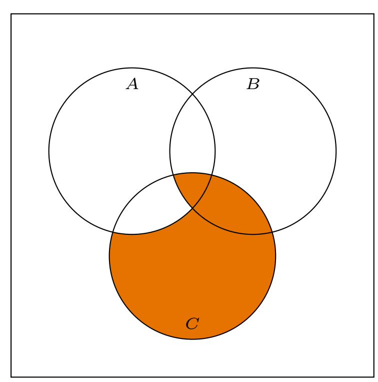




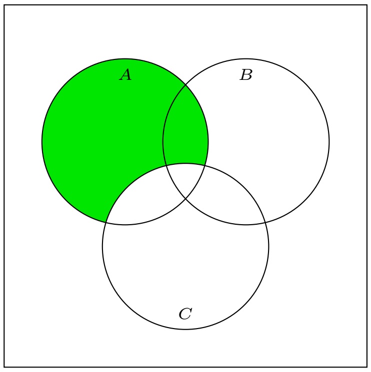




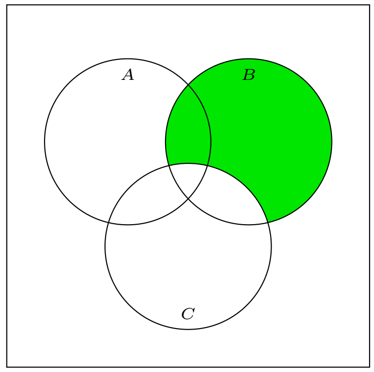




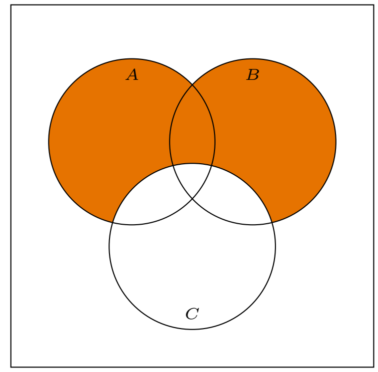
(a) The Venn Diagram in **Row 3 Column 3** represents the expression: $(\mathbf{A} \setminus \mathbf{C}) \cup (\mathbf{B} \setminus \mathbf{C})$.



$A \setminus C$

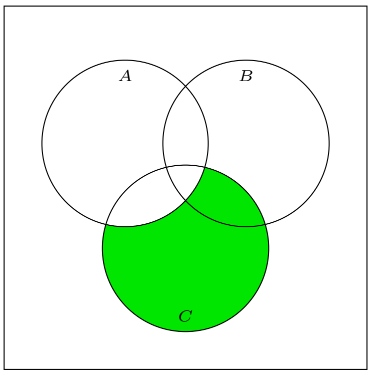


$B \setminus C$

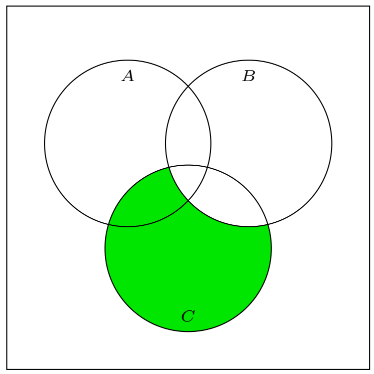


$(A \setminus C) \cup (B \setminus C)$

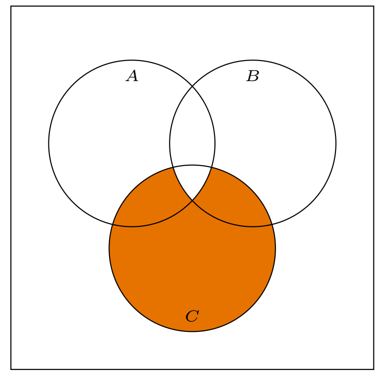
(b) The Venn Diagram in **Row 2 Column 1** represents the expression: $(\mathbf{C} \setminus \mathbf{A}) \cup (\mathbf{C} \setminus \mathbf{B})$.



$C \setminus A$

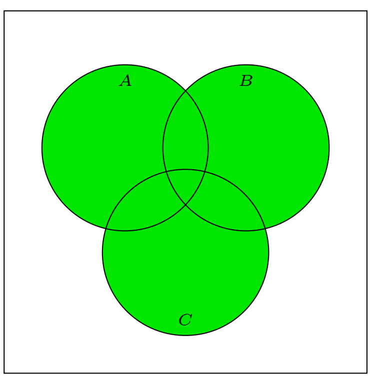


$C \setminus B$

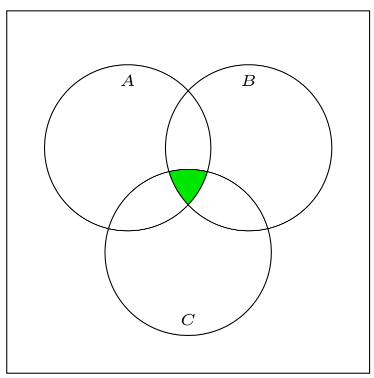


$(C \setminus A) \cup (C \setminus B)$

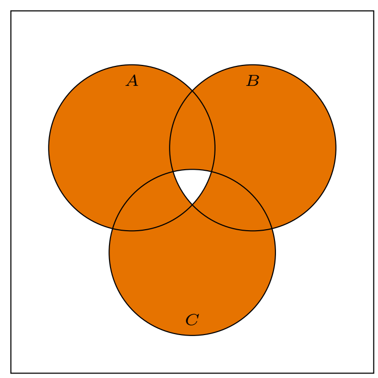
(c) The Venn Diagram in **Row 1 Column 2** represents the expression: $(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) \setminus (\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})$.



$A \cup B \cup C$

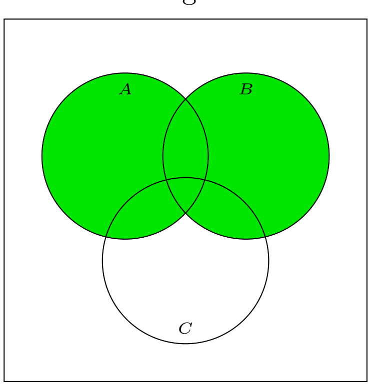


$A \cap B \cap C$

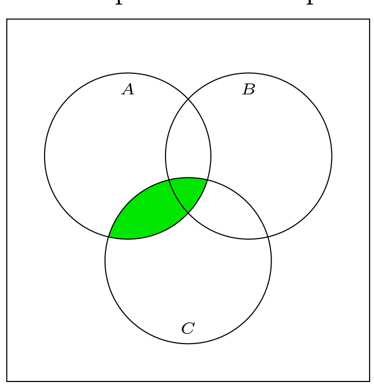


$(A \cup B \cup C) \setminus (A \cap B \cap C)$

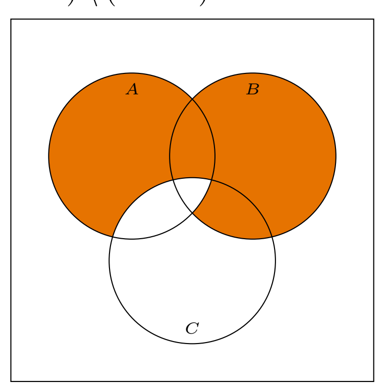
(d) The Venn Diagram in **Row 4 Column 1** represents the expression: $(\mathbf{A} \cup \mathbf{B}) \setminus (\mathbf{A} \cap \mathbf{C})$.



$A \cup B$



$A \cap C$



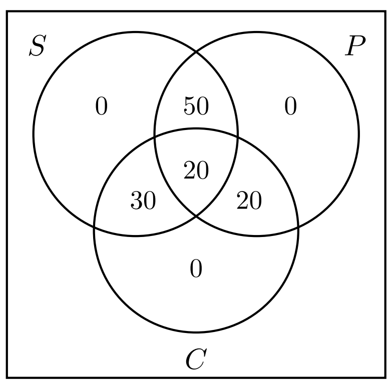
$(A \cup B) \setminus (A \cap C)$

3. At a winter party, every student from the Red club arrived with either a Red shirt, or Red pants, or a Red coat (these students may arrived with more than one Red item of clothing). Based on the data below, determine the total number of students from the Red club who attended the party.
   - None of the students attended the party with less than two Red items of clothing.
   - 70 students showed up with a Red shirt and Red pants (and maybe also with a Red coat).
   - 50 students showed up with a Red shirt and a Red coat (and maybe also with Red pants)
   - 40 students showed up with Red pants and a Red coat (and maybe also with a Red shirt).
   - 20 students showed up with all of the three Red items of clothings.

   Answer: 120.

   Notations: Let S be the set of students who attended the party with a Red shirt, let P be the set of students who attended the party with Red pants, and let C be the set of students who attended the party with a Red coat.

   Proof using a Venn diagram: The Venn diagram below reflects the data above.
   - Since each student who attended the party showed up with at least two Red items of clothing, it follows that $|S \cap \overline{P} \cap \overline{C}| = |\overline{S} \cap P \cap \overline{C}| = |\overline{S} \cap \overline{P} \cap C| = 0$.
   - Since 70 students showed up with a Red shirt and Red pants while 20 students showed up with all of the three Red items of clothing, it follows that 50 students showed up with a Red shirt and Red pants but without a Red coat: $|S \cap P \cap \overline{C}| = 70 - 20 = 50$.
   - Since 50 students showed up with a Red shirt and a Red coat while 20 students showed up with all of the three Red items of clothing, it follows that 30 students showed up with a Red shirt and a Red coat but without Red pants: $|S \cap \overline{P} \cap C| = 50 - 20 = 30$.
   - Since 40 students showed up with Red pants and a Red coat while 20 students showed up with all of the three Red items of clothing, it follows that 20 students showed up with Red pants and a Red coat but without a Red shirt: $|\overline{S} \cap P \cap C| = 40 - 20 = 20$.



   As a result, the total number of students from the Red club who attended the party is $120 = 50+30+20+20$.

   Proof using the Principle of Inclusion Exclusion:
   - Applying the principle of inclusion exclusion on the set S implies that
   $$|S| = |S \cap P| + |S \cap C| - |S \cap P \cap C| = 70 + 50 - 20 = 100$$
   - Applying the principle of inclusion exclusion on the set P implies that
   $$|P| = |P \cap S| + |P \cap C| - |S \cap P \cap C| = 70 + 40 - 20 = 90$$
   - Applying the principle of inclusion exclusion on the set C implies that
   $$|C| = |C \cap S| + |C \cap P| - |S \cap P \cap C| = 50 + 40 - 20 = 70$$
   - Applying the principle of inclusion exclusion on the set $S \cup P \cup C$ implies that
   $$\begin{aligned} |S \cup P \cup C| &= |S| + |P| + |C| - |S \cap P| - |S \cap C| - |P \cap C| + |S \cup P \cup C| \\ &= 100 + 90 + 70 - 70 - 50 - 40 + 20 = 120 \end{aligned}$$

At a winter party, every student from the Blue club arrived with either a Blue shirt, or Blue pants, or a Blue coat (these students may arrived with more than one Blue item of clothing). Based on the data below, determine the number of students from the Blue club who attended the party with all three Blue items of clothing.

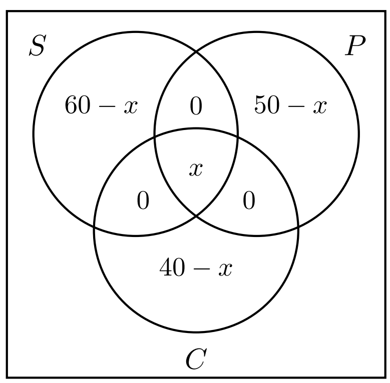
- In total, 130 students attended the party.
- None of them wore exactly two Blue items of clothing.
- 60 students showed up with a Blue shirt (and maybe also with Blue Pants and a Blue coat).
- 50 students showed up with Blue pants (and maybe also with a Blue shirt and a Blue coat).
- 40 students showed up with a Blue coat (and maybe also with a Blue shirt and Blue Pants).

Answer: **10**.

Notations: Let S be the set of students who attended the party with a Blue shirt, let P be the set of students who attended the party with Blue pants, and let C be the set of students who attended the party with a Blue coat. Let $x = |S \cap P \cap C|$ denote the number of students who attended the party with all three Blue items of clothing.

Proof using a Venn diagram: The Venn diagram below reflects the data above.

- Since no student who attended the party showed up with exactly two Blue items of clothing, it follows that $|S \cap P \cap \overline{C}| = |S \cap \overline{P} \cap C| = |\overline{S} \cap P \cap C| = 0$.
- Since 60 students showed up with a Blue shirt while x students showed up with all of the three Blue items of clothing, it follows that $60 - x$ students showed up with a Blue shirt but with neither Blue pants nor a Blue coat: $|S \cap \overline{P} \cap \overline{C}| = 60 - x$.
- Since 50 students showed up with Blue pants while x students showed up with all of the three Blue items of clothing, it follows that $50 - x$ students showed up with Blue pants but with neither a Blue shirt nor a Blue coat: $|\overline{S} \cap P \cap \overline{C}| = 50 - x$.
- Since 40 students showed up with a Blue coat while x students showed up with all of the three Blue items of clothing, it follows that $40 - x$ students showed up with a Blue coat but with neither a Blue shirt nor Blue pants: $|\overline{S} \cap \overline{P} \cap C| = 40 - x$.



Since in total 130 students attended the party, it follows that $130 = (60-x)+(50-x)+(40-x)+x = 150-2x$. This identity is equivalent to $x = (150 - 130)/2 = 10$.

Proof using the Principle of Inclusion Exclusion: Since no student who attended the party showed up with exactly two Blue items of clothing, it follows that

$$|S \cap P| = |S \cap C| = |P \cap C| = |S \cap P \cap C| = x$$

The principle of inclusion exclusion for the set $S \cup P \cup C$ is

$$|S \cup P \cup C| = |S| + |P| + |C| - |S \cap P| - |S \cap C| - |P \cap C| + |S \cap P \cap C|$$

Plugging the data and computed numbers for the sizes of the sets in the above identity implies that

$$130 = 60 + 50 + 40 - x - x - x + x = 150 - 2x$$

The above identity is equivalent to $x = (150 - 130)/2 = 10$.

4. **Proposition 1:** $L = z \wedge \neg(x \wedge y)$ is not equivalent to $R = z \vee \neg(x \vee y)$.

Proof: Either of the following two counterexamples is sufficient to prove that L and R are not equivalent.

(a) Consider the assignment $x = y = z = T$. It follows that $x \wedge y$ is True, which implies that $\neg(x \wedge y)$ is False, which implies that $L = z \wedge \neg(x \wedge y)$ is False. But since $z = T$, it follows that R is True. Due to this assignment, L is not equivalent to R.

(b) Consider the assignment $x = y = z = F$. It follows that $x \vee y$ is False, which implies that $\neg(x \vee y)$ is True, which implies that $R = z \vee \neg(x \vee y)$ is True. But since $z = F$, it follows that L is False. Due to this assignment, L is not equivalent to R.

Proof with truth tables: L and R are not equivalent because their respective truth tables (see below) have different final columns.

x	y	z	$x \wedge y$	$\neg(x \wedge y)$	$L = z \wedge \neg(x \wedge y)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	F	T	T
F	F	F	F	T	F

x	y	z	$x \vee y$	$\neg(x \vee y)$	$R = z \vee \neg(x \vee y)$
T	T	T	T	F	T
T	T	F	T	F	F
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	T	F	T
F	T	F	T	F	F
F	F	T	F	T	T
F	F	F	F	T	T

Remark: The truth tables for L and R confirm that there are no counterexamples other than the two identified in the first proof.

Proposition 2: $L = (z \vee y) \wedge (\neg z \vee \neg x)$ is equivalent to $R = (z \wedge \neg x) \vee (\neg z \wedge y)$.

Proof: The proposition follows because $L \equiv R$ when $z = T$ and when $z = F$.

(a) Assigning $z = T$ and $\neg z = F$ in L and R imply that

$$\begin{aligned} L = (z \vee y) \wedge (\neg z \vee \neg x) &= (T \vee y) \wedge (F \vee \neg x) = T \wedge \neg x = \neg x \\ R = (z \wedge \neg x) \vee (\neg z \wedge y) &= (T \wedge \neg x) \vee (F \wedge y) = \neg x \vee F = \neg x \end{aligned}$$

(b) Assigning $z = F$ and $\neg z = T$ in L and R imply that

$$\begin{aligned} L = (z \vee y) \wedge (\neg z \vee \neg x) &= (F \vee y) \wedge (T \vee \neg x) = y \wedge T = y \\ R = (z \wedge \neg x) \vee (\neg z \wedge y) &= (F \wedge \neg x) \vee (T \wedge y) = F \vee y = y \end{aligned}$$

Proof with truth tables: $L \equiv R$ because their respective truth tables (see below) have identical final columns.

x	y	z	$z \vee y$	$\neg z \vee \neg x$	$(z \vee y) \wedge (\neg z \vee \neg x)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	T	F

x	y	z	$z \wedge \neg x$	$\neg z \wedge y$	$(z \wedge \neg x) \vee (\neg z \wedge y)$
T	T	T	F	F	F
T	T	F	F	T	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	F	T
F	T	F	F	T	T
F	F	T	T	F	T
F	F	F	F	F	F

5. The table below shows which flowers are liked by three students. A T in the table means the student in that row likes the flower in that column. For example, the T in the (Bob, Iris) entry means Bob likes Iris while the F in the (Carol,Lily) entry means Carol does not like Lily.

Student \ Flowers	Daisy	Iris	Jasmine	Lily	Magnolia	Rose	Tulip
Alice	F	T	F	F	T	T	T
Bob	F	T	T	T	F	T	F
Carol	F	F	T	F	F	T	T

Let S be the set containing the 3 students and let F be the set containing the 7 flowers:

$$S = \{\text{Alice}, \text{Bob}, \text{Carol}\} \qquad F = \{\text{Daisy}, \text{Iris}, \text{Jasmine}, \text{Lily}, \text{Magnolia}, \text{Rose}, \text{Tulip}\}$$

(a) $\exists_{x\in S}\exists_{y\in F}\,(x \text{ likes } y)$ is **True** because there exists at least one T-entry in the table.
   - For example, Alice likes Iris.

(b) $\forall_{x\in S}\forall_{y\in F}\,(x \text{ likes } y)$ is **False** because there exists at least one F-entry in the table.
   - For example, Carol does not like Lily.

(c) $\exists_{x\in S}\forall_{y\in F}\,(x \text{ likes } y)$ is **False** because there exists at least one F-entry in each row of the table.
   - For example, Alice does not like Lily, Bob does not like Tulip, and Carol does not like Magnolia.

(d) $\exists_{y\in F}\forall_{x\in S}\,(x \text{ likes } y)$ is **True** because there exists at least one all T-entry column in the table.
   - For example, all three students like Rose.

(e) $\forall_{x\in S}\exists_{y\in F}\,(x \text{ likes } y)$ is **True** because there exists at least one T-entry in each row of the table.
   - For example, Alice likes Magnolia, Bob likes Jasmine, and Carol likes Tulip.

(f) $\forall_{y\in F}\exists_{x\in S}\,(x \text{ likes } y)$ is **False** because there exists at least one all F-entry column in the table.
   - For example, no student likes Daisy.

Remarks: Some answers imply other answers.

- If expression (b) is False, then expression (a) must be True. But if expression (a) is True, then expression (b) could be either True or False.
- If either expression (c) is False or expression (f) is False then expression (b) must be False. But if expression (b) is False, then both expression (c) and expression (f) could be either True or False.
- If expression (f) is False, then expression (c) must be False. But if expression (c) is False, then expression (f) could be either True or False.
- If expression (d) is True, then expression (e) must be True. But if expression (e) is True, then expression (f) could be either True or False.

6. You meet A, B, and C.
   - A claims that B is lying.
   - B claims that C is lying.
   - C claims that both A and B are lying.

   Who is lying and who is telling the truth?

   Answer: B is telling the truth while A and C are lying.

   Explanation I: Assume that A is telling the truth. This would mean that A's statement is true, making B a liar. If B is a liar, it implies that C is telling the truth. This creates a contradiction, because C states that A is lying, which goes against the assumption. Therefore, the assumption must be wrong and A is lying. A's false statement means that B is telling the truth, which in turn means that C must be lying. This holds, as C's statement that B is lying is indeed a lie.

   Explanation II: Assume that B is lying. This would mean that C is telling the truth (based on B's statement), and that A is also telling the truth (based on A's statement). This scenario is impossible because C's truthful statement claims that A is lying, creating a contradiction. Since the initial assumption is false, the opposite must be true: B is telling the truth. Consequently, both A and C must be liars, because each of their statements falsely claims that B is lying.

   Explanation III: Assume that C is telling the truth. Then C's statement that A is lying must be true. However, both A and C claim that B is lying which imply that A is not a liar. Therefore, the initial assumption is false, and C must be lying. From this, B's statement must be true, making B a truth-teller. This confirms that A, who falsely claims that B is lying, is indeed a liar.

   Remark: A proof is incomplete if it only shows that one truth assignemt to the statements of A, B, and C works. For instance, that B's statement is true while A's and C's statements are false. To be complete, the proof must also demonstrate that this truth assignment is the only one possible.