CISC 2210 TR11 – Introduction to Discrete Structures

Midterm 2 Exam – Solutions

Nov 11, 2025

Problem 1: Induction

Prove by induction the following identity for all integers $n \geq 2$:

$$1+2+3+\cdots+(n-2)+(n-1)+n+(n-1)+(n-2)+\cdots+3+2+1 = n^2$$

In other words, using induction, prove for all integers $n \geq 2$ that n^2 is the sum of the positive integers ascending from 1 to n and then descending back to 1.

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= 1+2+3+\cdots+(n-2)+(n-1)+n+(n-1)+(n-2)+\cdots+3+2+1 \\ R(n) &= n^2 \end{aligned}$$

- *Induction base.* Prove that $L(2) = R(2)$:

$$L(2) = 1+2+1 = 4 = 2^2 = R(2)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 2$:

$$1+2+3+\cdots+(k-2)+(k-1)+k+(k-1)+(k-2)+\cdots+3+2+1 = k^2$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 2$:

$$\begin{aligned} L(k+1) &= 1+2+3+\cdots+(k-1)+k+(k+1)+k+(k-1)+\cdots+3+2+1 \\ &= L(k)+(k+1)+k \\ &= R(k)+2k+1 \\ &= k^2+2k+1 \\ &= (k+1)^2 \\ &= R(k+1) \end{aligned}$$

A direct proof: The proof applies the known identity $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ for $k = n$ and for $k = n-1$.

$$\begin{aligned} 1+2+\cdots+(n-1)+n+(n-1)+\cdots+2+1 &= \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i \\ &= \frac{n(n+1)}{2} + \frac{(n-1)n}{2} \\ &= \frac{n(n+1)+(n-1)n}{2} \\ &= \frac{n(n+1+n-1)}{2} \\ &= \frac{n(2n)}{2} \\ &= n^2 \end{aligned}$$

Problem 2: Recursion

Define the following recursive formula $T(n)$ for all nonnegative integers $n \geq 0$:

$$T(n) = \begin{cases} 1 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 2T(n-1) + 3T(n-2) & \text{for } n \geq 2 \end{cases}$$

Problem 2a: Find a closed-form expression for $T(n)$.

Bottom-Up evaluation:

$$\begin{aligned} T(0) &= 1 = 3^0 \\ T(1) &= 3 = 3^1 \\ T(2) &= 2T(1) + 3T(0) = 2 \cdot 3 + 3 \cdot 1 = 6 + 3 = 9 = 3^2 \\ T(3) &= 2T(2) + 3T(1) = 2 \cdot 9 + 3 \cdot 3 = 18 + 9 = 27 = 3^3 \\ T(4) &= 2T(3) + 3T(2) = 2 \cdot 27 + 3 \cdot 9 = 54 + 27 = 81 = 3^4 \\ &\vdots \qquad\qquad \vdots \\ T(n) &= 3^n \end{aligned}$$

Problem 2b: Prove that the expression from part (a) is correct.

Proof by induction: $T(n) = 3^n$ for $n \geq 0$.

- *Induction base.* $T(0) = 1 = 3^0$ and $T(1) = 3 = 3^1$ for $n = 0$ and $n = 1$.
- *Induction hypothesis.* Assume $T(n-1) = 3^{n-1}$ and $T(n-2) = 3^{n-2}$ for $n > 1$.
- *Inductive step.* Prove $T(n) = 3^n$ for $n > 1$.

$$\begin{aligned} T(n) &= 2T(n-1) + 3T(n-2) \\ &= 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} \\ &= 2 \cdot 3^{n-1} + 3^{n-1} \\ &= 3 \cdot 3^{n-1} \\ &= 3^n \end{aligned}$$

Problem 3: Counting

A ternary string is a list with repetitions $(d_1, d_2, \ldots, d_n)$ such that $d_i \in \{0, 1, 2\}$ for all $1 \le i \le n$. For example, 11021 is a length-5 ternary string, 0010 is a length-4 ternary string, and 22 is a length-2 ternary string.

Problem 3a: For $n \ge 1$, how many length-n ternary strings are there?

Answer: $\mathbf{3^n}$.

Explanation: Since each of the n positions in a length-n ternary string can be filled in 3 different ways, it follows that there are 3^n possible length-n ternary strings.

Problem 3b: For $n \ge 2$, in how many length-n ternary strings $d_1 = d_n$?

Answer: $\mathbf{3^{n-1}}$.

Explanation: There are 3 choices for each of the first $n-1$ positions (from d_1 to d_{n-1}). The final position, d_n, is then uniquely determined by the choice for d_1, leaving only 1 option. It follows that there are $3^{n-1} \cdot 1 = 3^{n-1}$ length-n ternary strings satisfying the condition $d_1 = d_n$.

Problem 3c: For $n \ge 2$, in how many length-n ternary strings $d_1 \ne d_n$?

Answer: $\mathbf{2 \cdot 3^{n-1} = 3^n - 3^{n-1}}$.

Explanation 1: There are 3 choices for each of the first $n-1$ positions (from d_1 to d_{n-1}). The final position, d_n, then has only 2 available choices, as it must be different from d_1. It follows that there are $3^{n-1} \cdot 2 = 2 \cdot 3^{n-1}$ length-n ternary strings satisfying the condition $d_1 \ne d_n$.

Explanation 2: For $n \ge 2$, for any ternary string, the first and last positions are either equal ($d_1 = d_n$) or different ($d_1 \ne d_n$). These two cases are mutually exclusive and cover all possibilities. Therefore, the sum of the counts for these two cases — the answers to part (b) and part (c) — must equal the total number of length-n ternary strings from part (a). It follows that there are $3^n - 3^{n-1}$ length-n ternary strings satisfying the condition $d_1 \ne d_n$.

Problem 3d: For $n \ge 1$, how many length-n ternary strings do not contain 0?

Answer: $\mathbf{2^n}$.

Explanation: Since each of the n positions in a length-n ternry string that does not contain 0 can be filled only in 2 different ways (either 1 or 2), it follows that there are 2^n possible length-n ternary strings that do not contain 0.

Problem 3e: For $n \ge 1$, how many length-n ternary strings contain at least one 0?

Answer: $\mathbf{3^n - 2^n}$.

Explanation: For $n \ge 1$, any ternary string either does not contain 0 or must contain at least one 0. These two cases are mutually exclusive and cover all possibilities. Therefore, the sum of the counts for these two cases — the answers to part (d) and part (e) — must equal the total number of length-n ternary strings from part (a). It follows that there are $3^n - 2^n$ length-n ternary strings that contain at least one 0.

Problem 3f: For $n \ge 2$, in how many length-n ternary strings $d_i \ne d_{i-1}$ for all $2 \le i \le n$?

Answer: $\mathbf{3 \cdot 2^{n-1}}$.

Explanation: There are 3 choices for the first position (d_1). After this position is determined, there are only 2 choices available for each of the remaining $n-1$ positions (from d_2 to d_n), since each of these positions must be different from its immediate predecessor. It follows that there are $3 \cdot 2^{n-1}$ length-n ternary strings satisfying the condition $d_i \ne d_{i-1}$ for all $2 \le i \le n$.

Problem 4: Combinatorics

A 10-member club needs to form two separate 3-person committees.

Problem 4a: How many ways can the club select the two committees if members are allowed to be on both?

Answer: $\mathbf{14440 = \binom{10}{3} \times \binom{10}{3}}$.

Explanation: First, there are $\binom{10}{3}$ ways to select the 3 members of the first committee from the 10 club members. Then, there are again $\binom{10}{3}$ ways to select the 3 members of the second committee, since the choice for the second committee is independent of the first and club members may serve on both. It follows that the number of ways to select the two 3-person committees is $\binom{10}{3} \times \binom{10}{3}$.

Evaluation: $\mathbf{\binom{10}{3} \times \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \times \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120 \times 120 = 14400}$.

Problem 4b: How many ways can the club select the two committees if the committees must be disjoint (meaning no member can be on both)?

Answer: $\mathbf{4200 = \binom{10}{3} \times \binom{7}{3} = \binom{10}{6} \times \binom{6}{3}}$.

Explanation 1: First, there are $\binom{10}{3}$ ways to select the 3 members of the first committee from the 10 club members. Then, since club members may not serve on both committees, the 3 members for the second committee must be chosen from the remaining 7 members, leaving $\binom{7}{3}$ ways. It follows that the number of ways to select the two disjoint 3-person committees is $\binom{10}{3} \times \binom{7}{3}$.

Evaluation 1: $\mathbf{\binom{10}{3} \times \binom{7}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \times \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 120 \times 35 = 4200}$.

Explanation 2: First, there are $\binom{10}{6}$ ways to select the 6 members who will serve on one of the two committees from the 10 club members. Then, there are $\binom{6}{3}$ ways to select who among these 6 will be on the first 3-person committee. Since the committees must be disjoint, the remaining 3 members are automatically assigned to the second committee. It follows that the number of ways to select the two disjoint 3-person committees is $\binom{10}{6} \times \binom{6}{3}$.

Evaluation 2: $\mathbf{\binom{10}{6} \times \binom{6}{3} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \times \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 210 \times 20 = 4200}$.

Problem 4c: How many ways can the club select the two committees if they must share exactly one common member?

Answer: $\mathbf{7560 = 10 \times \binom{9}{2} \times \binom{7}{2} = 10 \times \binom{9}{4} \times \binom{4}{2} = \binom{10}{5} \times 5 \times \binom{4}{2} = \binom{10}{3} \times 3 \times \binom{7}{2}}$.

Definition: Call the one person who will serve on both committees the *president*.

Explanation 1: First, there are 10 ways to select the president. Then, there are $\binom{9}{2}$ ways to select the 2 other club members who will join the president in the first committee. Finally, there are $\binom{7}{2}$ ways to select the 2 club members who will join the president in the second committee, chosen from the 7 club members not yet assigned to the first committee. It follows that the number of ways to select the two 3-person committees sharing exactly one person is $10 \times \binom{9}{2} \times \binom{7}{2}$.

Evaluation 1: $\mathbf{10 \times \binom{9}{2} \times \binom{7}{2} = 10 \times \frac{9 \cdot 8}{2 \cdot 1} \times \frac{7 \cdot 6}{2 \cdot 1} = 10 \times 36 \times 21 = 7560}$

Explanation 2: First, there are 10 ways to select the president. Then, from the remaining 9 club members, select the 4 other members who will complete the two committees, which can be done in $\binom{9}{4}$ ways. Finally, from this group of 4, choose 2 to join the president in the first committee (leaving the other 2 to join the president in the second committee), which can be done in $\binom{4}{2}$ ways. It follows that the number of ways to select the two 3-person committees sharing exactly one person is $10 \times \binom{9}{4} \times \binom{4}{2}$.

Evaluation 2: $\mathbf{10 \times \binom{9}{4} \times \binom{4}{2} = 10 \times \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \times \frac{4 \cdot 3}{2 \cdot 1} = 10 \times 126 \times 6 = 7560}$

Explanation 3: First, there are $\binom{10}{5}$ ways to select the 5 club members who will be involved in forming the two committees. Then, from this group of 5, there are 5 ways to select the president. Finally, from the remaining 4 members in this group, choose 2 to join the president in the first committee (which automatically leaves the other 2 to join the president in the second committee), and this can be done in $\binom{4}{2}$ ways. It follows that the number of ways to select the two 3-person committees sharing exactly one person is $\mathbf{\binom{10}{5} \times 5 \times \binom{4}{2}}$.

Evaluation 3: $\mathbf{\binom{10}{5} \times 5 \times \binom{4}{2} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \times 5 \times \frac{4 \cdot 3}{2 \cdot 1} = 252 \times 5 \times 6 = 7560}$

Explanation 4: First, there are $\binom{10}{3}$ ways to select the 3 club members to serve on the first committee. Then, from this group of 3, there are 3 ways to select the president. Finally, there are $\binom{7}{2}$ ways to select the 2 other members to join the president in the second committee, chosen from the 7 club members not on the first committee. It follows that the number of ways to select the two 3-person committees sharing exactly one person is $\mathbf{\binom{10}{3} \times 3 \times \binom{7}{2}}$.

Evaluation 4: $\mathbf{\binom{10}{3} \times 3 \times \binom{7}{2} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \times 3 \times \frac{7 \cdot 6}{2 \cdot 1} = 120 \times 3 \times 21 = 7560}$

Problem 5: Probability

Bag A contains 3 yellow and 9 red marbles, while Bag B contains 6 yellow and 6 blue marbles. A 6-sided fair die is rolled to determine which bag to draw from: if the die shows a 5 or 6 (a 1/3 probability), a single marble is blindly drawn from Bag A; otherwise (a 2/3 probability), a single marble is blindly drawn from Bag B.

An equivalent setting: This setting involves three bags: A, B1, and B2. Bag A contains 3 yellow and 9 red marbles while Bags B1 and B2 each contains 6 yellow and 6 blue marbles. A bag is selected at random, with each having a 1/3 probability of being chosen. Then one marble is blindly drawn from the chosen bag. Informally, this selection process is equivalent to the original problem because selecting Bag A (with its 1/3 probability) "reflects" the case where the original Bag A was selected, while selecting either Bag B1 or Bag B2 (a combined 2/3 probability) "reflects" the case where the original Bag B was selected.

Problem 5a: What is the probability that the drawn marble is yellow?

Answer: 5/12.

A probability explanation: With probability 1/3, the marble is drawn from Bag A, and then with probability 3/12, this marble is yellow. Therefore, the probability of drawing a yellow marble from Bag A is $(1/3) \times (3/12) = 1/12$. Similarly, with probability 2/3, the marble is drawn from Bag B, and then with probability 6/12, this marble is yellow. This gives a probability of $(2/3) \times (6/12) = 1/3$ for drawing a yellow marble from Bag B. Combining both arguments, the probability to draw a yellow marble is $(1/12) + (1/3) = 5/12$.

A counting explanation: In the equivalent setting, the three bags contain 36 marbles in total, broken down into 12 blue, 9 red, and 15 yellow. Because each bag is chosen with a 1/3 probability and each contains 12 marbles, every individual marble has an equal 1/36 probability of being drawn. Consequently, the overall probability of drawing a yellow marble is the number of yellow marbles divided by the total number of marbles, which is $15/36 = 5/12$.

Problem 5b: What is the probability that Bag A was the chosen bag, given that the drawn marble was yellow?

Answer: 1/5.

A probability explanation: Let A be the event that Bag A was chosen, and let Y be the event that the drawn marble was yellow. The goal is to compute the conditional probability $p(A|Y)$. In Part (a), it was established that the probability of the joint event $A \cap Y$ is $p(A \cap Y) = 1/12$, and the probability of the event Y is $p(Y) = 5/12$. By the definition of conditional probability, it follows that

$$p(A|Y) = \frac{p(A \cap Y)}{p(Y)} = \frac{1/12}{5/12} = \frac{1}{5}$$

A counting explanation: In the equivalent setting, the three bags contain 15 yellow marbles in total. Of these, 3 yellow marbles "represent" the 3 yellow marbles from the original Bag A, while the other 12 yellow marbles "represent" the 6 yellow marbles from the original Bag B . Since it is given that the drawn marble was yellow, the sample space is reduced to just these 15 possible yellow outcomes. As a result, the probability that Bag A was the chosen bag, given that the drawn marble was yellow is $3/15 = 1/5$.