CISC 2210 TR2 – Introduction to Discrete Structures

Midterm 2 Exam

Nov 11, 2025

Id: ....

Problem	Maximum Points	Your Points
Induction	100	
Recursion	100	
Counting	100	
Combinatorics	100	
Probability	100	

Structure, problem selection, and credit:

- You have 2 hours to complete the exam.
- There are 5 problems. Each problem is a "mini-exam" by itself with a 5% weight in the final grade for the class. However, the grade of each individual problem counts only if it is higher than the final exam grade.

  Strategy: It is better to try first answering the questions relating to topics you have mastered. Note that since there is no cumulative grade, one fully correct answer is better than two or more partially correct answers.
- You will get only partial credit if you fail to justify or prove your answers. You will get 20% of the credit for any problem or part of a problem if you leave the allocated space for the answer empty. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam **by themselves** without any external help from other people, the Internet, books, notes, or calculators. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.

Problem 1: Induction (100 credits)

Prove by induction the following identity for all integers $n \geq 2$:

$$n+(n-1)+(n-2)+\cdots+3+2+1+2+3+\cdots+(n-2)+(n-1)+n = n^2+n-1$$

In other words, using induction, prove for all integers $n \geq 2$ that n^2+n-1 is the sum of the positive integers descending from n to 1 and then ascending back to n.

Problem 2: Recursion (100 credits)

Define the following recursive formula $T(n)$ for all positive integers $n \geq 1$:

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ 4 & \text{for } n = 2 \\ T(n-1) + 2T(n-2) & \text{for } n \geq 2 \end{cases}$$

Problem 2a (50 credits)

Find a closed-form expression for $T(n)$. Justify your answer.

Problem 2b (50 credits)

Prove that the expression from part (a) is correct. Note that bottom-up or top-down evaluations will not be considered valid proofs.

Problem 3: Counting (100 credits)

An XYZ-word is a list with repetitions $(w_1, w_2, \ldots, w_n)$ such that $w_i \in \{X, Y, Z\}$ for all $1 \leq i \leq n$. For example, XXYZX is a length-5 XYZ-word, ZZZX is a length-4 XYZ-word, and YY is a length-2 XYZ-word.

Justify your answers to the following six questions.

Problem 3a (10 credits)

For $n \geq 1$, how many length-n XYZ-words are there?

Problem 3b (20 credits)

For $n \geq 2$, in how many length-n XYZ-words $w_1 = w_n$?

Problem 3c (20 credits)

For $n \geq 2$, in how many length-n XYZ-words $w_1 \neq w_n$?

Problem 3d (20 credits)

For $n \geq 1$, how many length-n XYZ-words do not contain Z?

Problem 3e (20 credits)

For $n \geq 1$, how many length-n XYZ-words contain at least one Z?

Problem 3f (10 credits)

For $n \geq 2$, in how many length-n XYZ-words $w_i \neq w_{i-1}$ for all $2 \leq i \leq n$?

Problem 4: Combinatorics (100 credits)

From a 12-player roster, a coach selects 5 starters and 2 co-captains.

Justify your answers for parts (a), (b), and (c). In your answers you do not need to evaluate binomial coefficients. You may leave them as $\binom{n}{k}$ for some n and k.

Problem 4a (40 credits)

How many ways can the coach select 5 starters from the 12 players, and also select 2 co-captains from the entire 12-player roster?

Problem 4b (40 credits)

How many ways can the coach select the 5 starters and 2 co-captains, given that the co-captains must be two players from the starting 5?

Problem 4c (20 credits)

How many ways can the coach first select 5 starters, and then select one co-captain from the 5 starters and the other co-captain from the 7 non-starters?

Problem 5: Probability (100 credits)

A bag initially contains 3 blue and 6 red marbles. A 6-sided fair die is rolled: if the die shows a 1 or 2 (a 1/3 probability), one blue marble is added to the bag; otherwise (a 2/3 probability), one red marble is added to the bag. After the new marble is in the bag, one marble is blindly drawn from the bag.

Justify your answers for Part (a) and Part (b).

Problem 5a (50 credits)

What is the probability that the drawn marble is blue?

Problem 5b (50 credits)

What is the probability that a blue marble was added to the bag, given that the drawn marble was blue?