CISC 2210 TR2 – Introduction to Discrete Structures

Midterm 2 Exam – Solutions

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Problem 1: Induction

Prove by induction the following identity for all integers $n \geq 2$:

$$n+(n-1)+(n-2)+\cdots+3+2+1+2+3+\cdots+(n-2)+(n-1)+n = n^2+n-1$$

In other words, using induction, prove for all integers $n \geq 2$ that n^2+n-1 is the sum of the positive integers descending from n to 1 and then ascending back to n.

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= n+(n-1)+(n-2)+\cdots+3+2+1+2+3+\cdots+(n-2)+(n-1)+n \\ R(n) &= n^2+n-1 \end{aligned}$$

- *Induction base.* Prove that $L(2) = R(2)$:

$$L(2) = 2+1+2 = 5 = 2^2+2-1 = R(2)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 2$:

$$k+(k-1)+(k-2)+\cdots+3+2+1+2+3+\cdots+(k-2)+(k-1)+k = k^2+k-1$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 2$:

$$\begin{aligned} L(k+1) &= (k+1)+k+(k-1)+\cdots+3+2+1+2+3+\cdots+(k-1)+k+(k+1) \\ &= L(k)+2(k+1) \\ &= R(k)+2k+2 \\ &= k^2+k-1+2k+2 \\ &= (k^2+2k+1)+k \\ &= (k+1)^2+(k+1)-1 \\ &= R(k+1) \end{aligned}$$

A direct proof: The proof applies the known identity $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

$$\begin{aligned} n+(n-1)+(n-2)+\cdots+3+2+1+2+3+\cdots+(n-2)+(n-1)+n &= 2\sum_{i=1}^{n} i - 1 \\ &= 2\frac{n(n+1)}{2} - 1 \\ &= n(n+1) - 1 \\ &= n^2+n-1 \end{aligned}$$

Problem 2: Recursion

Define the following recursive formula $T(n)$ for all positive integers $n \geq 1$:

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ 4 & \text{for } n = 2 \\ T(n-1) + 2T(n-2) & \text{for } n \geq 2 \end{cases}$$

Problem 2a: Find a closed-form expression for $T(n)$.

Bottom-Up evaluation:

$$\begin{aligned}
T(1) &= 2 = 2^1 \\
T(2) &= 4 = 2^2 \\
T(3) &= T(2) + 2T(1) = 4 + 2 \cdot 2 = 4 + 4 = 8 = 2^3 \\
T(4) &= T(3) + 2T(2) = 8 + 2 \cdot 4 = 8 + 8 = 16 = 2^4 \\
T(5) &= T(4) + 2T(3) = 16 + 2 \cdot 8 = 16 + 16 = 32 = 2^5 \\
&\vdots \qquad\qquad \vdots \\
T(n) &= 2^n
\end{aligned}$$

Problem 2b: Prove that the expression from part (a) is correct.

Proof by induction: $T(n) = 2^n$ for $n \geq 1$.

- *Induction base.* $T(1) = 2 = 2^1$ and $T(2) = 4 = 2^2$ for $n = 1$ and $n = 2$.
- *Induction hypothesis.* Assume $T(n-1) = 2^{n-1}$ and $T(n-2) = 2^{n-2}$ for $n > 2$.
- *Inductive step.* Prove $T(n) = 2^n$ for $n > 2$.

$$\begin{aligned}
T(n) &= T(n-1) + 2T(n-2) \\
&= 2^{n-1} + 2 \cdot 2^{n-2} \\
&= 2^{n-1} + 2^{n-1} \\
&= 2 \cdot 2^{n-1} \\
&= 2^n
\end{aligned}$$

Problem 3: Counting

An XYZ-word is a list with repetitions $(w_1, w_2, \ldots, w_n)$ such that $w_i \in \{X, Y, Z\}$ for all $1 \leq i \leq n$. For example, XXYZX is a length-5 XYZ-word, ZZZX is a length-4 XYZ-word, and YY is a length-2 XYZ-word.

Problem 3a: For $n \geq 1$, how many length-n XYZ-words are there?

Answer: $\mathbf{3^n}$.

Explanation: Since each of the n positions in a length-n XYZ-word can be filled in 3 different ways, it follows that there are 3^n possible length-n XYZ-words.

Problem 3b: For $n \geq 2$, in how many length-n XYZ-words $w_1 = w_n$?

Answer: $\mathbf{3^{n-1}}$.

Explanation: There are 3 choices for each of the first $n-1$ positions (from w_1 to w_{n-1}). The final position, w_n, is then uniquely determined by the choice for w_1, leaving only 1 option. It follows that there are $3^{n-1} \cdot 1 = 3^{n-1}$ length-n XYZ-words satisfying the condition $w_1 = w_n$.

Problem 3c: For $n \geq 2$, in how many length-n XYZ-words $w_1 \neq w_n$?

Answer: $\mathbf{2 \cdot 3^{n-1} = 3^n - 3^{n-1}}$.

Explanation 1: There are 3 choices for each of the first $n-1$ positions (from w_1 to w_{n-1}). The final position, w_n, then has only 2 available choices, as it must be different from w_1. It follows that there are $3^{n-1} \cdot 2 = 2 \cdot 3^{n-1}$ length-n XYZ-words satisfying the condition $w_1 \neq w_n$.

Explanation 2: For $n \geq 2$, for any XYZ-word, the first and last positions are either equal ($w_1 = w_n$) or different ($w_1 \neq w_n$). These two cases are mutually exclusive and cover all possibilities. Therefore, the sum of the counts for these two cases — the answers to part (b) and part (c) — must equal the total number of length-n XYZ-words from part (a). It follows that there are $3^n - 3^{n-1}$ length-n XYZ-words satisfying the condition $w_1 \neq w_n$.

Problem 3d: For $n \geq 1$, how many length-n XYZ-words do not contain Z?

Answer: $\mathbf{2^n}$.

Explanation: Since each of the n positions in a length-n XYZ-word that does not contain Z can be filled only in 2 different ways (either X or Y), it follows that there are 2^n possible length-n XYZ-words that do not contain Z.

Problem 3e: For $n \geq 1$, how many length-n XYZ-words contain at least one Z?

Answer: $\mathbf{3^n - 2^n}$.

Explanation: For $n \geq 1$, any XYZ-word either does not contain Z or must contain at least one Z. These two cases are mutually exclusive and cover all possibilities. Therefore, the sum of the counts for these two cases — the answers to part (d) and part (e) — must equal the total number of length-n XYZ-words from part (a). It follows that there are $3^n - 2^n$ length-n XYZ-words that contain at least one Z.

Problem 3f: For $n \geq 2$, in how many length-n XYZ-words $w_i \neq w_{i-1}$ for all $2 \leq i \leq n$?

Answer: $\mathbf{3 \cdot 2^{n-1}}$.

Explanation: There are 3 choices for the first position (w_1). After this position is determined, there are only 2 choices available for each of the remaining $n-1$ positions (from w_2 to w_n), since each of these positions must be different from its immediate predecessor. It follows that there are $3 \cdot 2^{n-1}$ length-n XYZ-words satisfying the condition $w_i \neq w_{i-1}$ for all $2 \leq i \leq n$.

Problem 4: Combinatorics

From a 12-player roster, a coach selects a 5 starters and 2 co-captains.

Problem 4a:

How many ways can the coach select 5 starters from the 12 players, and also select 2 co-captains from the entire 12-player roster?

Answer: $\mathbf{52272} = \binom{\mathbf{12}}{\mathbf{5}} \times \binom{\mathbf{12}}{\mathbf{2}}$.

Explanation: First, there are $\binom{12}{5}$ ways to select the 5 starters from the 12 players. Since the 2 co-captains do not have to be starters, their selection is an independent choice from the entire 12-player roster, which can be done in $\binom{12}{2}$ ways. It follows that the number of ways to select the 5 starters and the 2 co-captains is $\binom{12}{5} \times \binom{12}{2}$.

Evaluation: $\binom{\mathbf{12}}{\mathbf{5}} \times \binom{\mathbf{12}}{\mathbf{2}} = \frac{\mathbf{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}}{\mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} \times \frac{\mathbf{12 \cdot 11}}{\mathbf{2 \cdot 1}} = \mathbf{792} \times \mathbf{66} = \mathbf{52272}$

Problem 4b:

How many ways can the coach select the 5 starters and 2 co-captains, given that the co-captains must be two players from the starting 5?

Answer: $\mathbf{7920} = \binom{\mathbf{12}}{\mathbf{5}} \times \binom{\mathbf{5}}{\mathbf{2}} = \binom{\mathbf{12}}{\mathbf{2}} \times \binom{\mathbf{10}}{\mathbf{3}}$.

Explanation 1: First, there are $\binom{12}{5}$ ways to select the 5 starters from the 12 players. Then, from that chosen group of 5 starters, there are $\binom{5}{2}$ ways to designate the 2 co-captains. It follows that the number of ways to select the 5 starters and 2 co-captains, while satisfying the condition that both co-captains must be starters, is $\binom{12}{5} \times \binom{5}{2}$.

Evaluation 1: $\binom{\mathbf{12}}{\mathbf{5}} \times \binom{\mathbf{5}}{\mathbf{2}} = \frac{\mathbf{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}}{\mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} \times \frac{\mathbf{5 \cdot 4}}{\mathbf{2 \cdot 1}} = \mathbf{792} \times \mathbf{10} = \mathbf{7920}$.

Explanation 2: First, there are $\binom{12}{2}$ ways to select the 2 co-captains from the 12 players. Then, from the remaining 10 players, there are $\binom{10}{3}$ ways to select the other 2 starters. It follows that the number of ways to select the 5 starters and 2 co-captains, while satisfying the condition that both co-captains must be starters, is $\binom{12}{1} \times \binom{10}{3}$.

Evaluation 2: $\binom{\mathbf{12}}{\mathbf{2}} \times \binom{\mathbf{10}}{\mathbf{3}} = \frac{\mathbf{12 \cdot 11}}{\mathbf{2 \cdot 1}} \times \frac{\mathbf{10 \cdot 9 \cdot 8}}{\mathbf{3 \cdot 2 \cdot 1}} = \mathbf{66} \times \mathbf{120} = \mathbf{7920}$.

Problem 4c:

How many ways can the coach first select 5 starters, and then select one co-captain from the 5 starters and the other co-captain from the 7 non-starters?

Answer: $\mathbf{27720} = \binom{\mathbf{12}}{\mathbf{5}} \times \mathbf{5} \times \mathbf{7} = \binom{\mathbf{12}}{\mathbf{2}} \times \mathbf{2} \times \binom{\mathbf{10}}{\mathbf{4}}$.

Explanation 1: First, there are $\binom{12}{5}$ ways to select the 5 starters from the 12 players. Then, there are 5×7 ways to select the 2 co-captains, since 1 co-captain must be chosen from the 5 starters and the other must be chosen from the 7 non-starters. It follows that the number of ways to select the 5 starters and 2 co-captains, while satisfying the condition that exactly one co-captain is a starter, is $\binom{12}{5} \times 5 \times 7$.

Evaluation 1: $\binom{\mathbf{12}}{\mathbf{5}} \times \mathbf{5} \times \mathbf{7} = \frac{\mathbf{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}}{\mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} \times \mathbf{5} \times \mathbf{7} = \mathbf{792} \times \mathbf{5} \times \mathbf{7} = \mathbf{27720}$.

Explanation 2: First, there are $\binom{12}{2}$ ways to select the 2 co-captains from the 12 players. Once these two are chosen, there are 2 ways to designate which one will be a starter and which one will be a non-starter. Finally, the remaining 4 starters must be selected from the 10 players who are not co-captains, which can be done in $\binom{10}{4}$ ways. It follows that the number of ways to select the 5 starters and 2 co-captains, satisfying the condition that exactly one co-captain is a starter, is $\binom{12}{2} \times 2 \times \binom{10}{4}$.

Evaluation 2: $\binom{\mathbf{12}}{\mathbf{2}} \times \mathbf{2} \times \binom{\mathbf{10}}{\mathbf{4}} = \frac{\mathbf{12 \cdot 11}}{\mathbf{2 \cdot 1}} \times \mathbf{2} \times \frac{\mathbf{10 \cdot 9 \cdot 8 \cdot 7}}{\mathbf{4 \cdot 3 \cdot 2 \cdot 1}} = \mathbf{66} \times \mathbf{2} \times \mathbf{210} = \mathbf{27720}$.

Problem 5: Probability

A bag initially contains 3 blue and 6 red marbles. A 6-sided fair die is rolled: if the die shows a 1 or 2 (a 1/3 probability), one blue marble is added to the bag; otherwise (a 2/3 probability), one red marble is added to the bag. After the new marble is in the bag, one marble is blindly drawn from the bag.

An equivalent setting: This setting involves three bags: A, B, and C. Bag A contains 4 blue and 8 red marbles while Bags B and C each contains 3 blue and 7 red marbles. A bag is selected at random, with each having a 1/3 probability of being chosen. Then one marble is blindly drawn from the chosen bag. Informally, this selection process is equivalent to the original problem because selecting Bag A (with its 1/3 probability) "reflects" the case where a blue marble was added, while selecting either Bag B or Bag C (a combined 2/3 probability) "reflects" the case where a red marble was added.

Problem 5a: What is the probability that the drawn marble is blue?

Answer: 1/3.

A probability explanation: Consider two mutually exclusive cases. First, with probability 1/3, a blue marble is added, resulting in a bag with 10 marbles, 4 of which are blue. The probability of then drawing a blue marble is 4/10, making the joint probability of this case (adding blue and drawing blue) $(1/3) \times (4/10) = 2/15$. Second, with probability 2/3, a red marble is added, resulting in a bag with 10 marbles, 3 of which are blue. The probability of then drawing a blue marble is 3/10, making the joint probability of this case (adding red and drawing blue) $(2/3) \times (3/10) = 1/5$. The overall probability of drawing a blue marble is the sum of these two case probabilities: $(2/15) + (1/5) = 1/3$.

A counting explanation: In the equivalent setting, the three bags contain 30 marbles in total, broken down into 10 blue and 20 red. Because each bag is chosen with a 1/3 probability and each contains 10 marbles, every individual marble has an equal 1/30 probability of being drawn. Consequently, the overall probability of drawing a blue marble is the number of blue marbles divided by the total number of marbles, which is $10/30 = 1/3$.

Problem 5b: What is the probability that a blue marble was added to the bag, given that the drawn marble was blue?

Answer: 2/5.

A probability explanation: Let A be the event that a blue marble was added to the bag, and let B be the event that a blue marble was drawn from the bag. The goal is to compute the conditional probability $p(A|B)$. In Part (a), it was established that the probability of the joint event $A \cap B$ is $p(A \cap B) = 2/15$, and the probability of event B is $p(B) = 1/3$. By the definition of conditional probability, it follows that

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{2/15}{1/3} = \frac{2}{5}$$

A counting explanation: In the equivalent setting, the three bags contain 10 blue marbles in total. Of these, 4 blue marbles "reflect" the "a blue marble was added to the bag" event, while the other 6 blue marbles "reflect" the "a red marble was added to the bag" event from the original setting. Since it is given that the drawn marble was blue, the sample space is reduced to just these 10 possible blue outcomes. As a result, the probability that a blue marble was added to the bag, given that the drawn marble was blue is $4/10 = 2/5$.